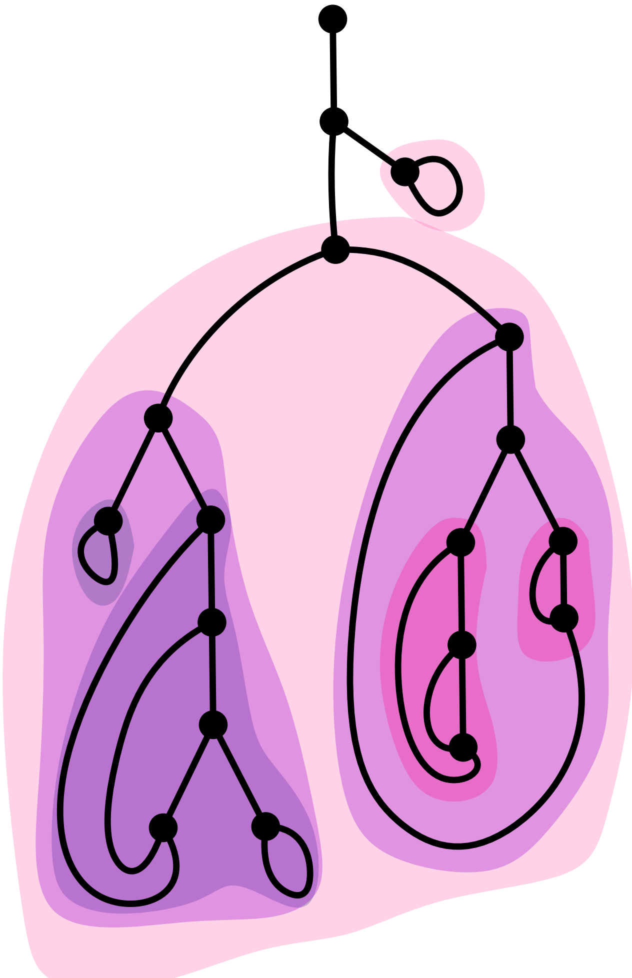
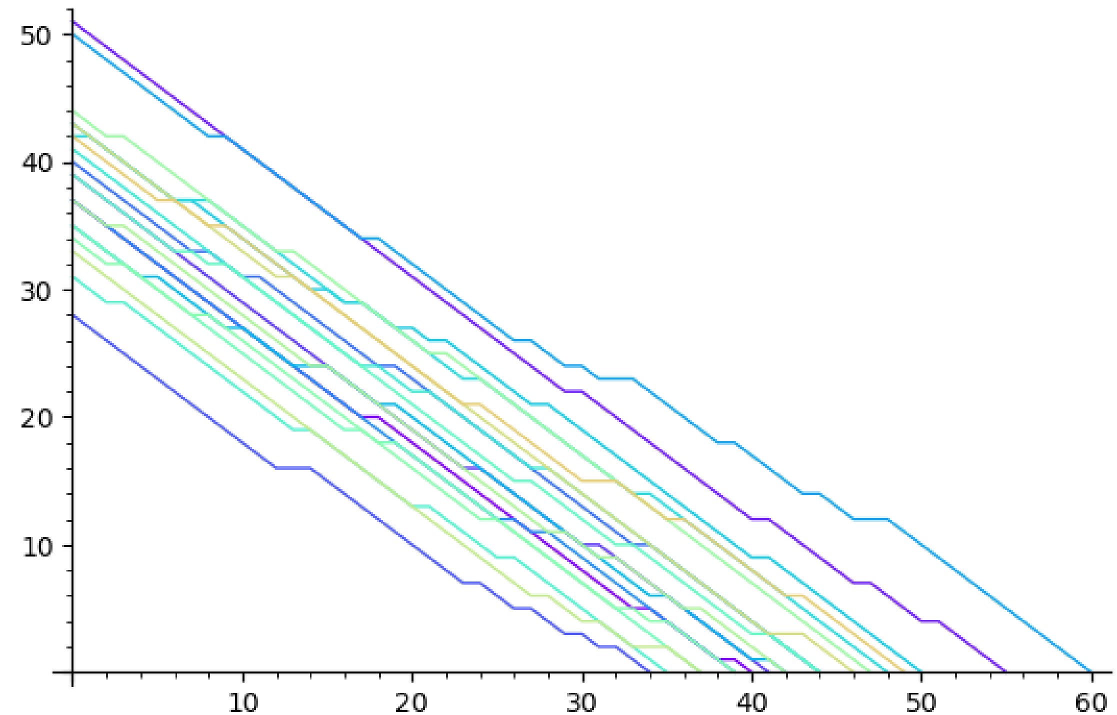
On the number of β-redices in random closed linear λ-terms





43rd Australasian Combinatorics Conference, 16 December 2021

Olivier Bodini (LIPN, Paris 13)

Alexandros Singh (LIPN, Paris 13)

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- We're interested in terms up to α-equivalence:

$$(\lambda x.xx)(\lambda x.xx) \overset{\alpha}{=} (\lambda y.yy)(\lambda x.xx) \overset{\alpha}{\neq} (\lambda y.ya)(\lambda x.xx)$$

Computing with the λ-calculus

- Substitution rule:

$$T_1[v := T_2]$$

"replace free occurences of v in T_1 with T_2"

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- Dynamics of the λ-calculus: β-reductions

(λ-terms together with β-reduction are enough to encode any computation!)

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  - Order in which redices are reduced matters!

$$(\lambda x.z)((\lambda x.(x\ x))(\lambda x.(x\ x))) \begin{cases} \to (\lambda x.z)((x\ x)[x := (\lambda x.(x\ x))]) = \ldots \\ \to z[x := (\lambda x.x\ x)(\lambda x.x\ x)] = z \end{cases}$$

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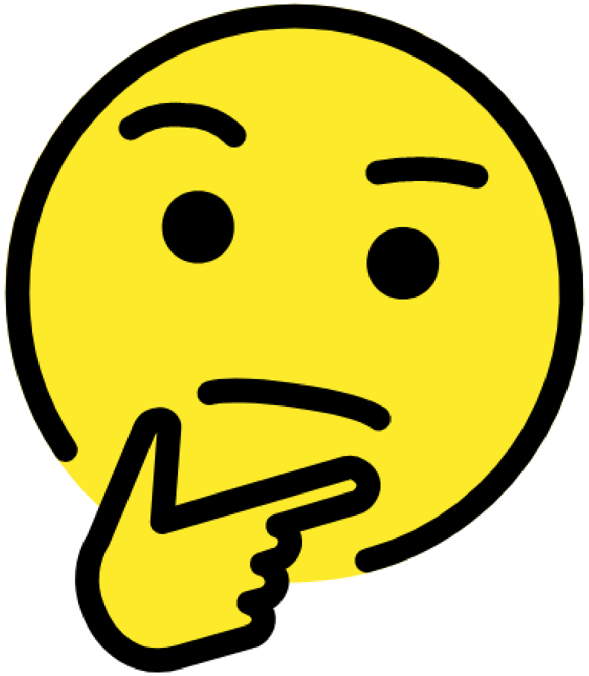

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  For terms expressed in the previously-presented syntax and size defined recursively as:

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  For terms expressed using de Bruijn indices or combinators (together with appropriate size functions)

Parameter sensitive to the definition of the syntax and the size of terms!

- Almost every simply-typed λ-term has a long β-reduction sequence [SAKT17]

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Linear Terms: bound variables occur **exactly** once

$\lambda x.\lambda y.(y\ x)a$

$\lambda x.\lambda y.(y\ a)(b\ x)$

$\lambda x.a(\lambda z.(\lambda y.y\ (x\ z)))$

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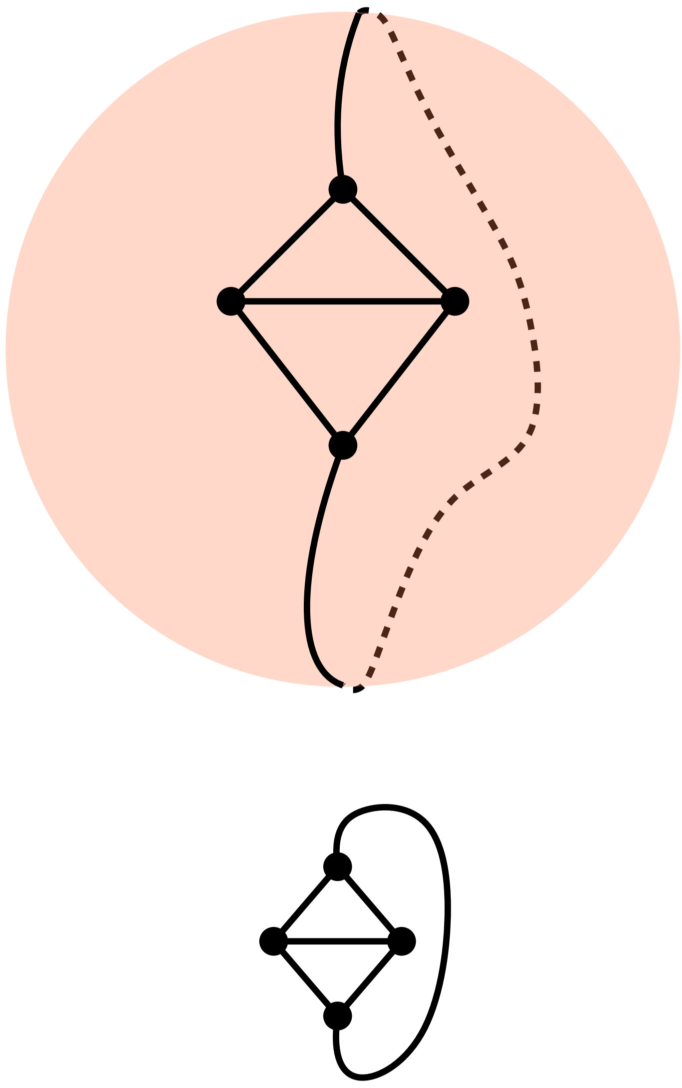
asymptotically!

seq. of random variables!

uniform distribution
on the set of terms of size n

What are maps?



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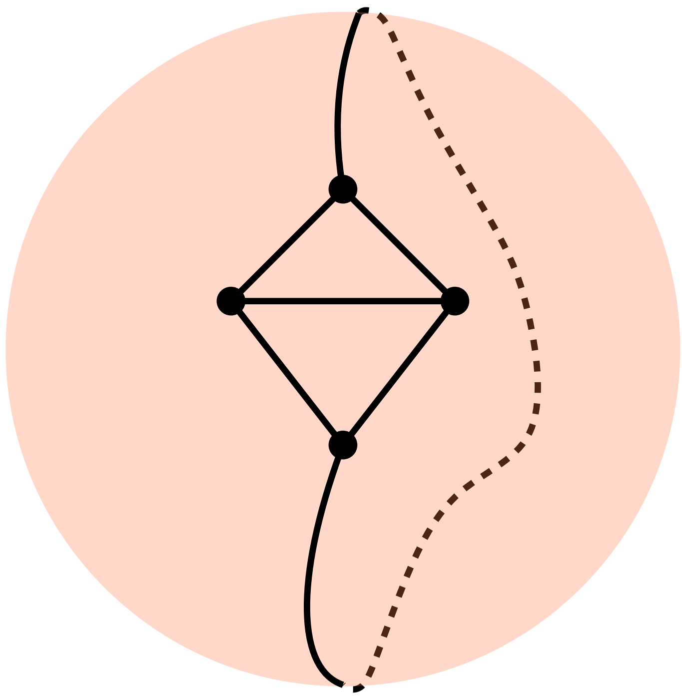

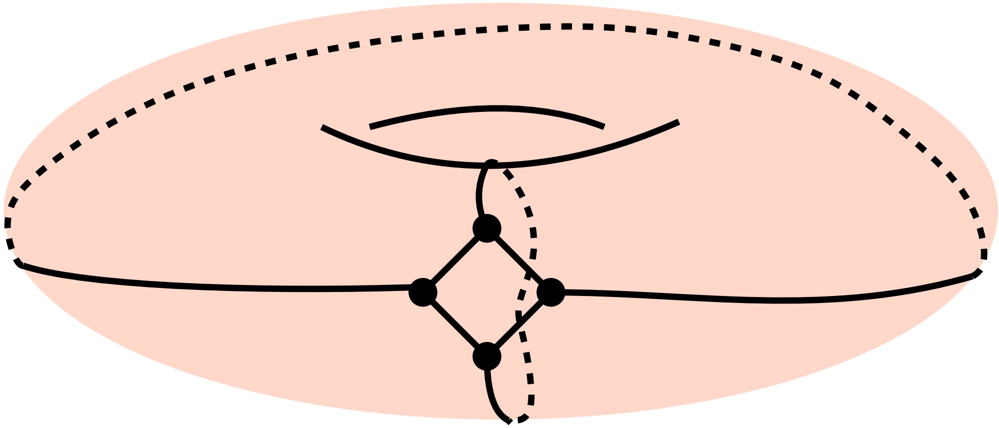

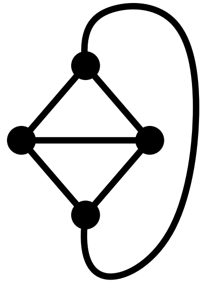

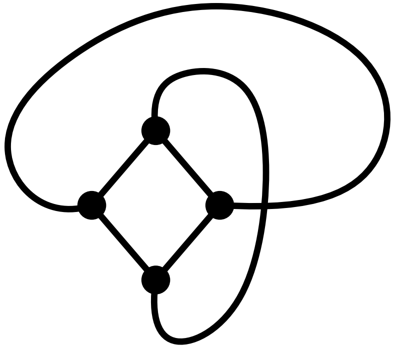

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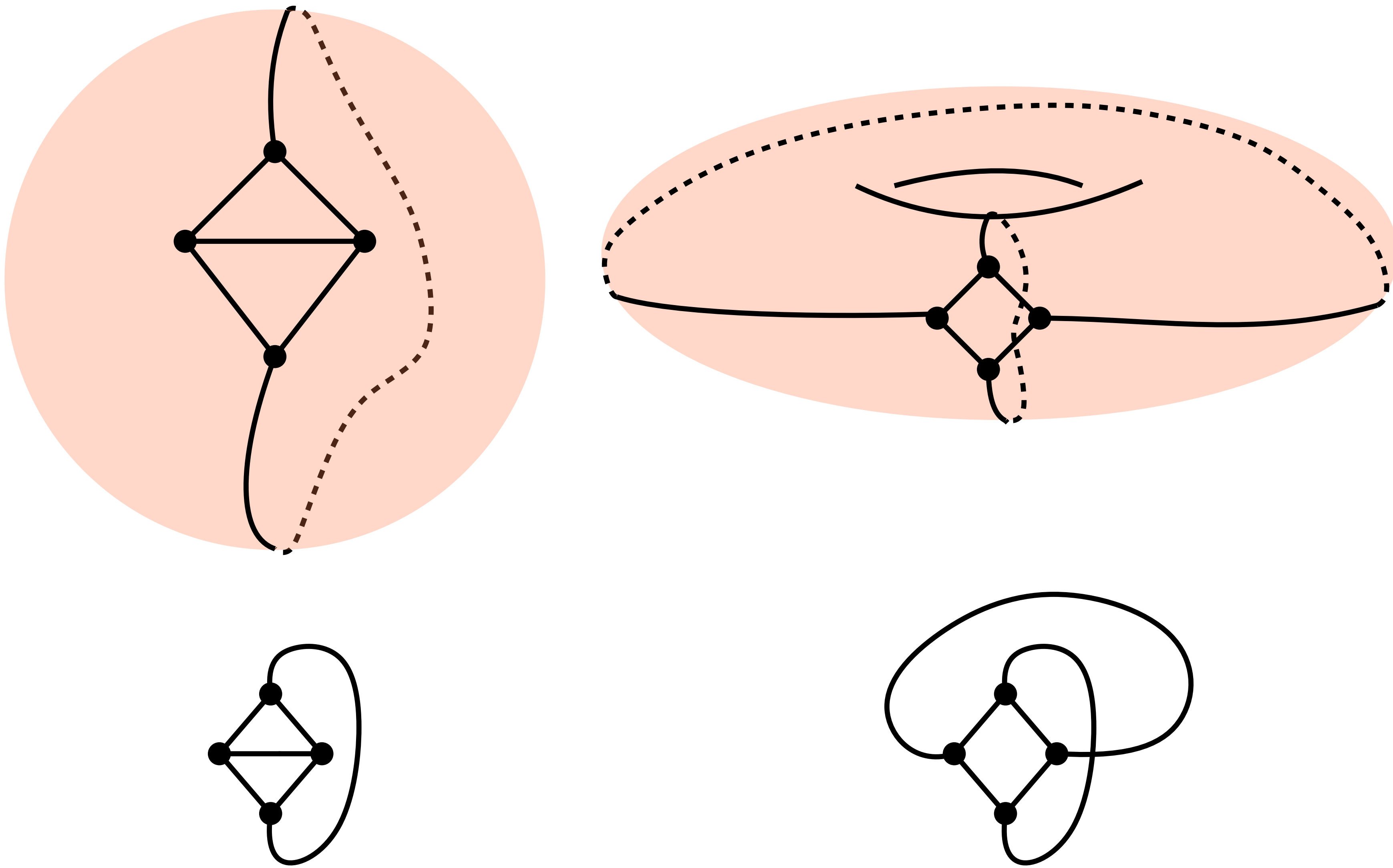
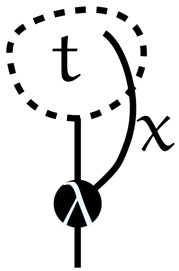


We're interested in unrestricted genus, restricted vertex degrees

Why should you, a logician, be interested in maps?

String diagrams! [BGJ13, Z16]

$\bullet = x$



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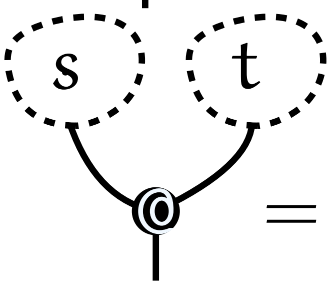

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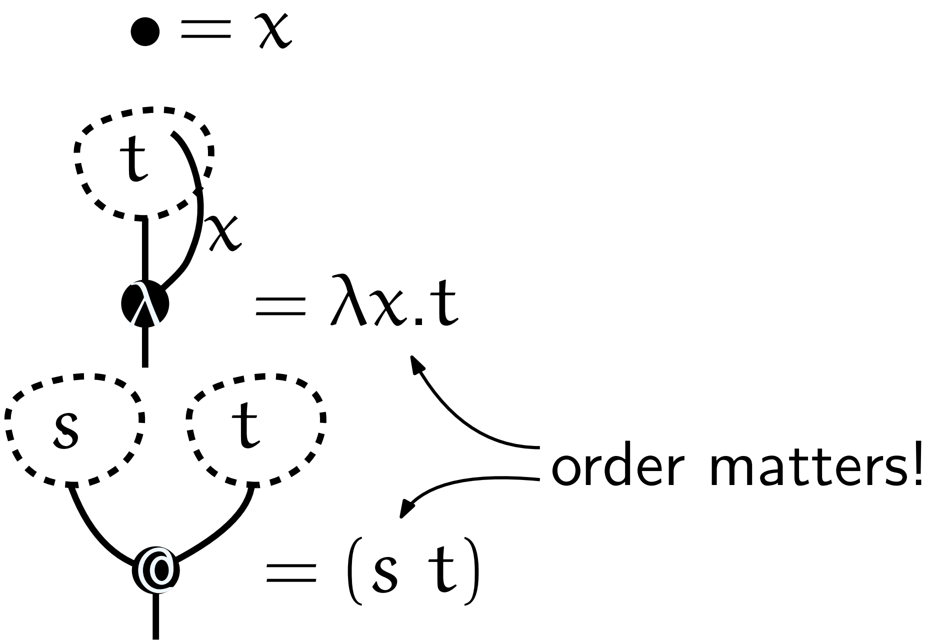

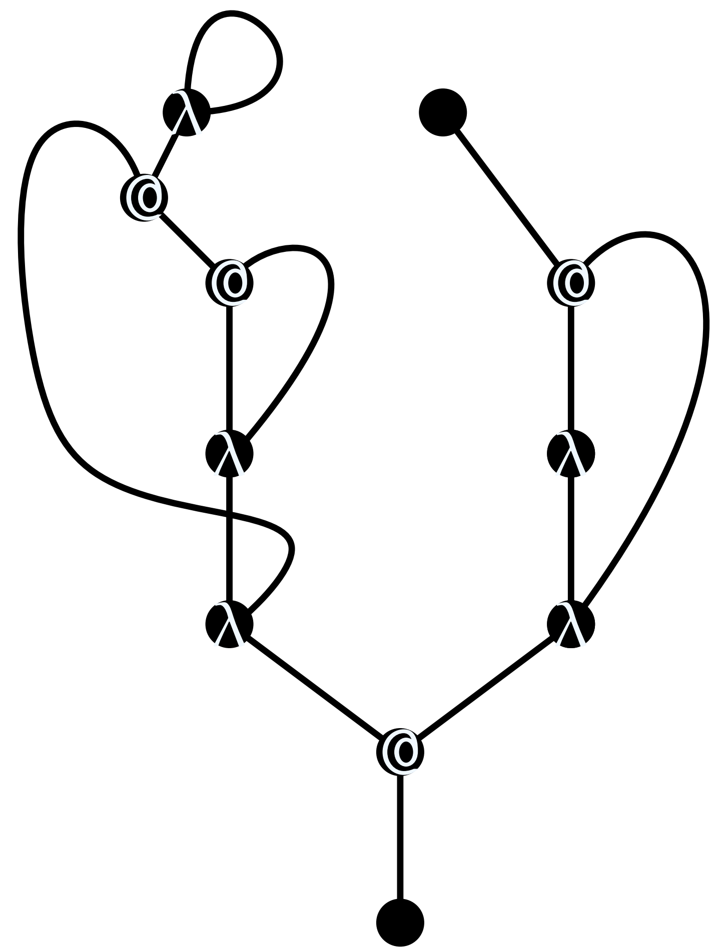

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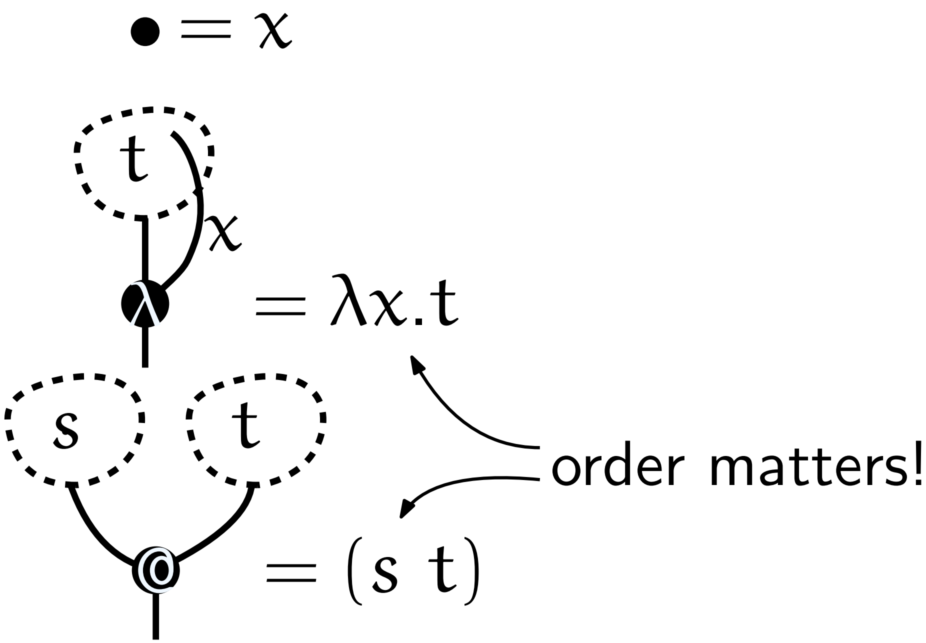

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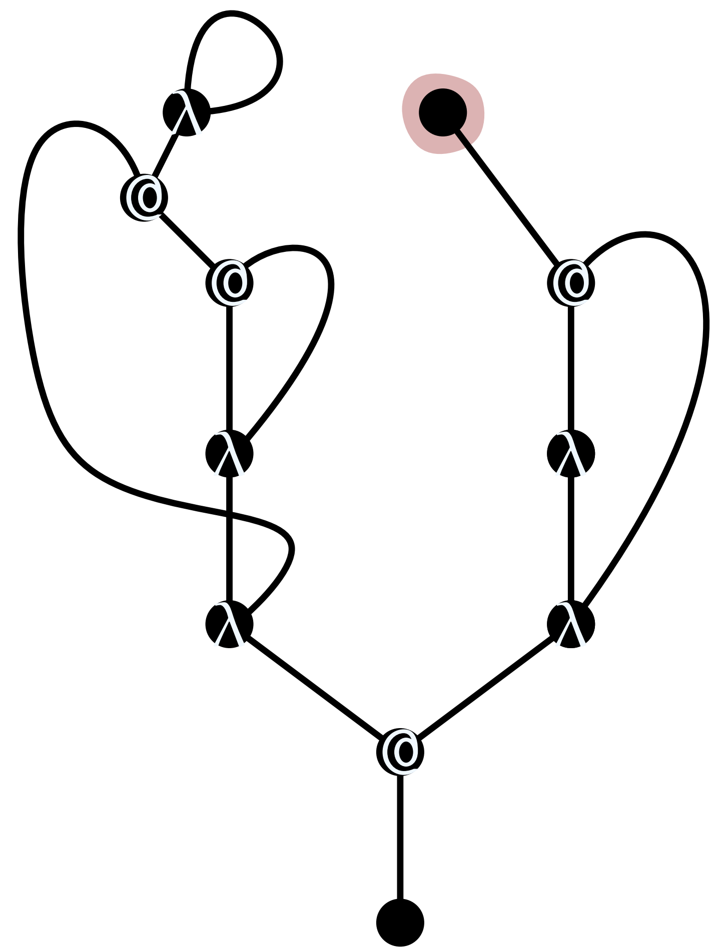

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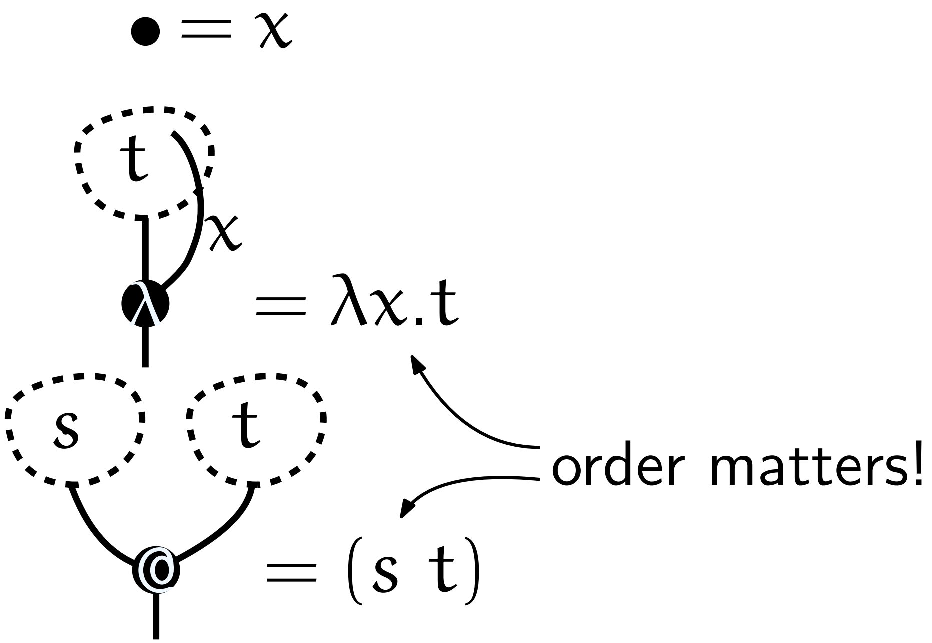

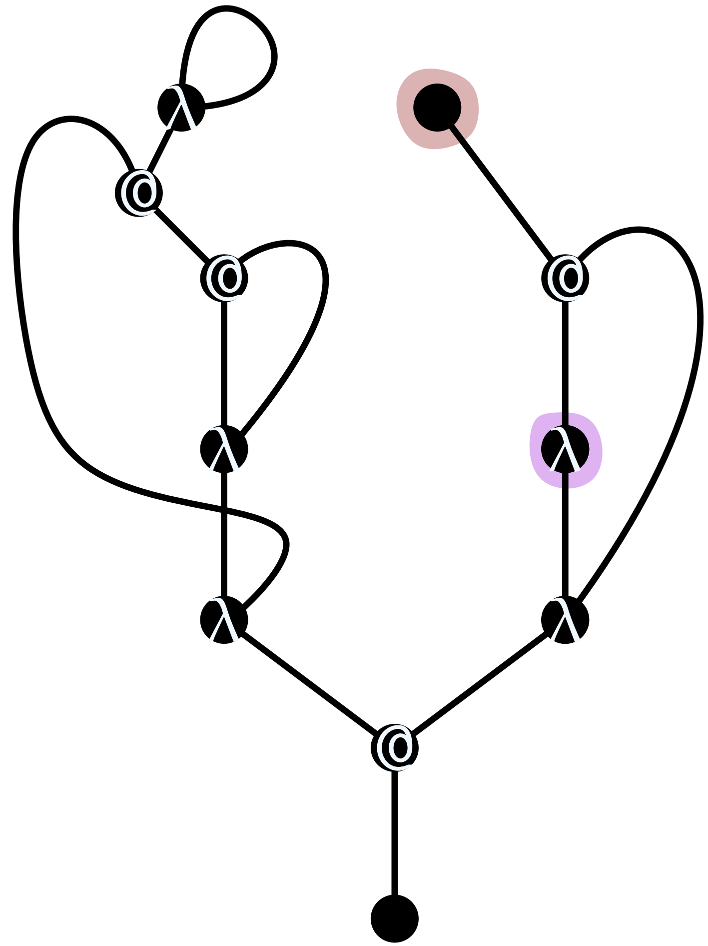

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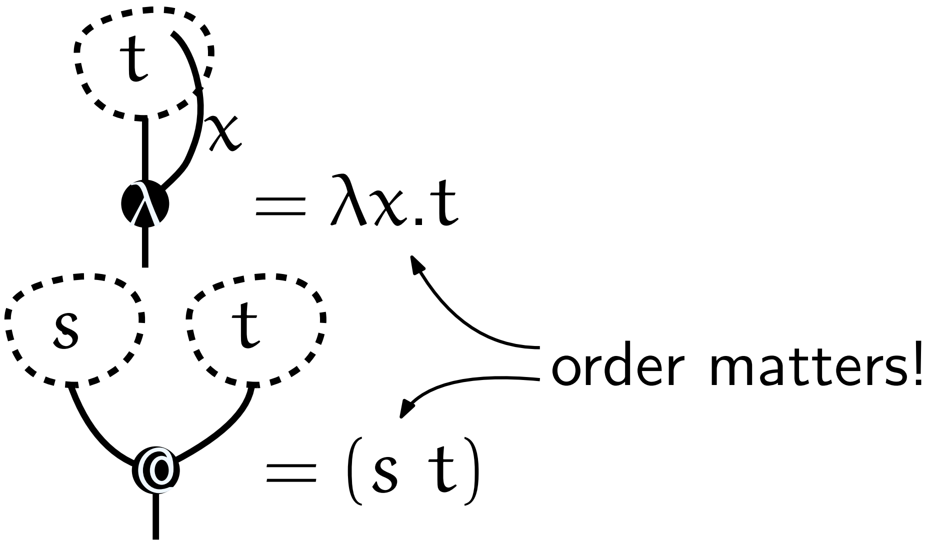

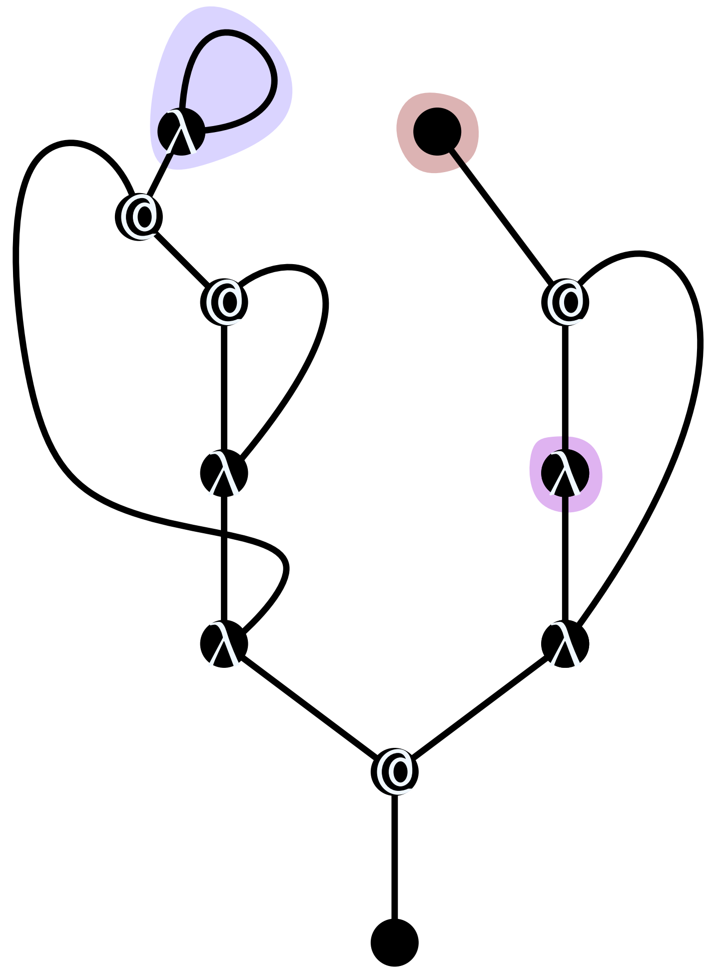

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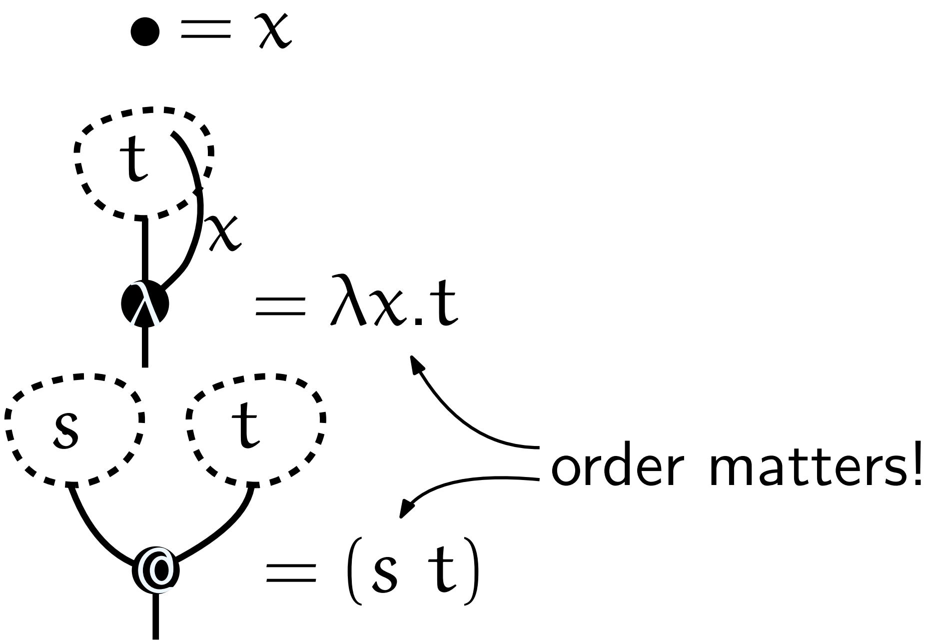

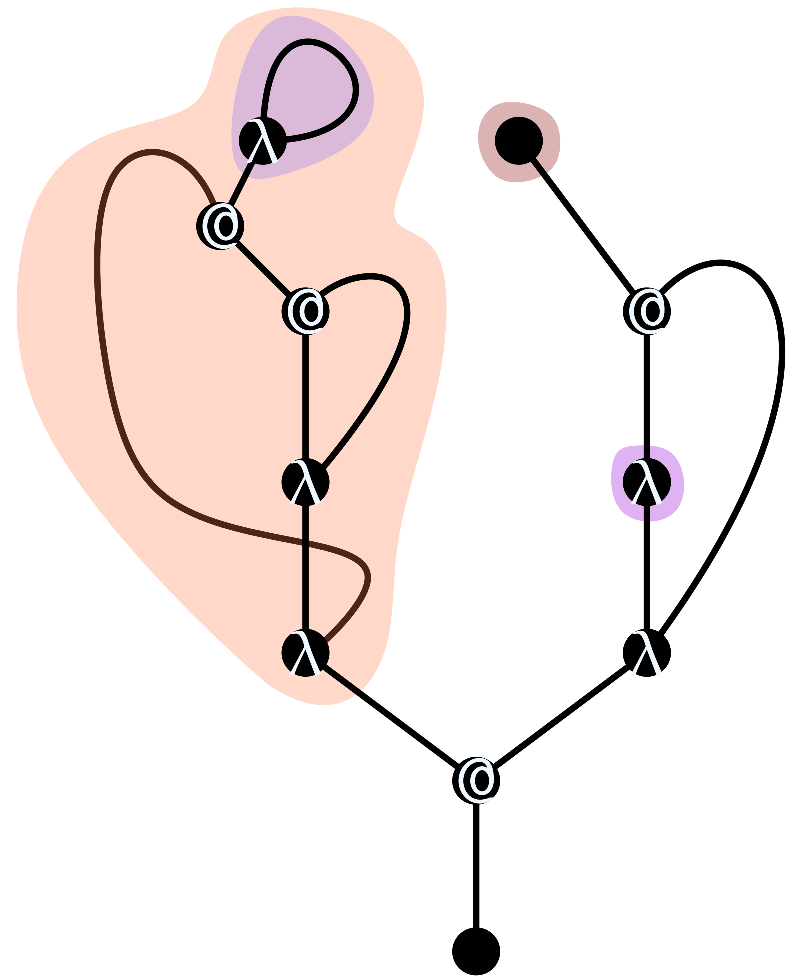

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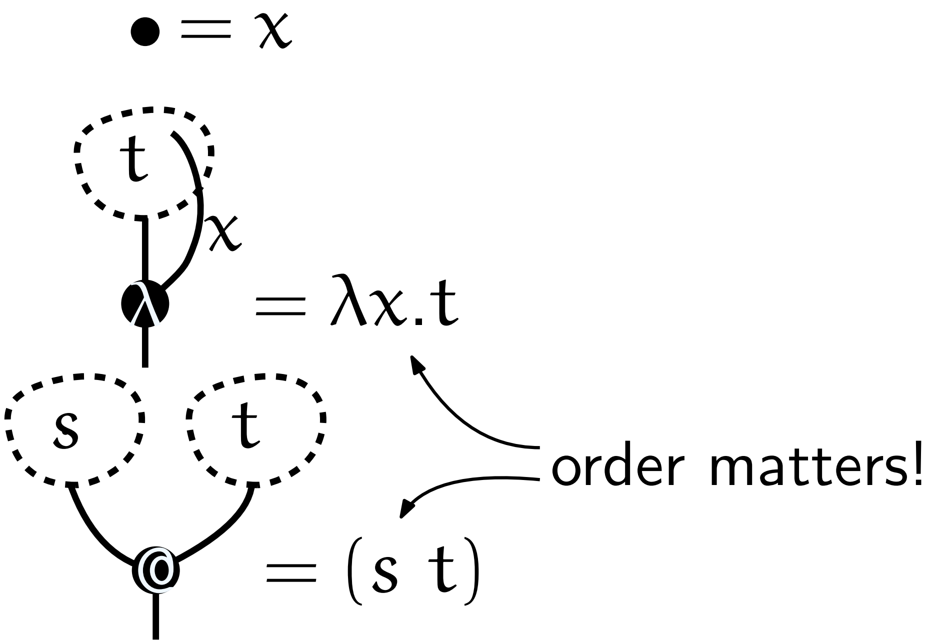

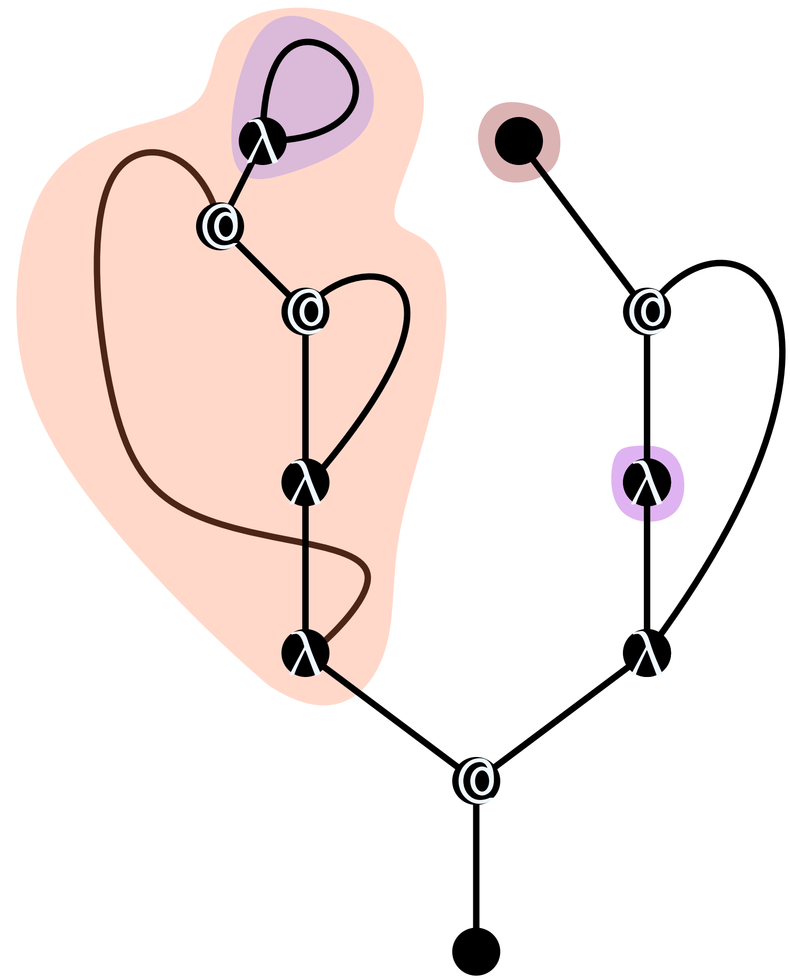

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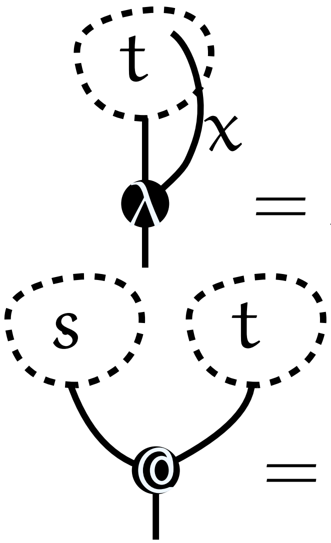
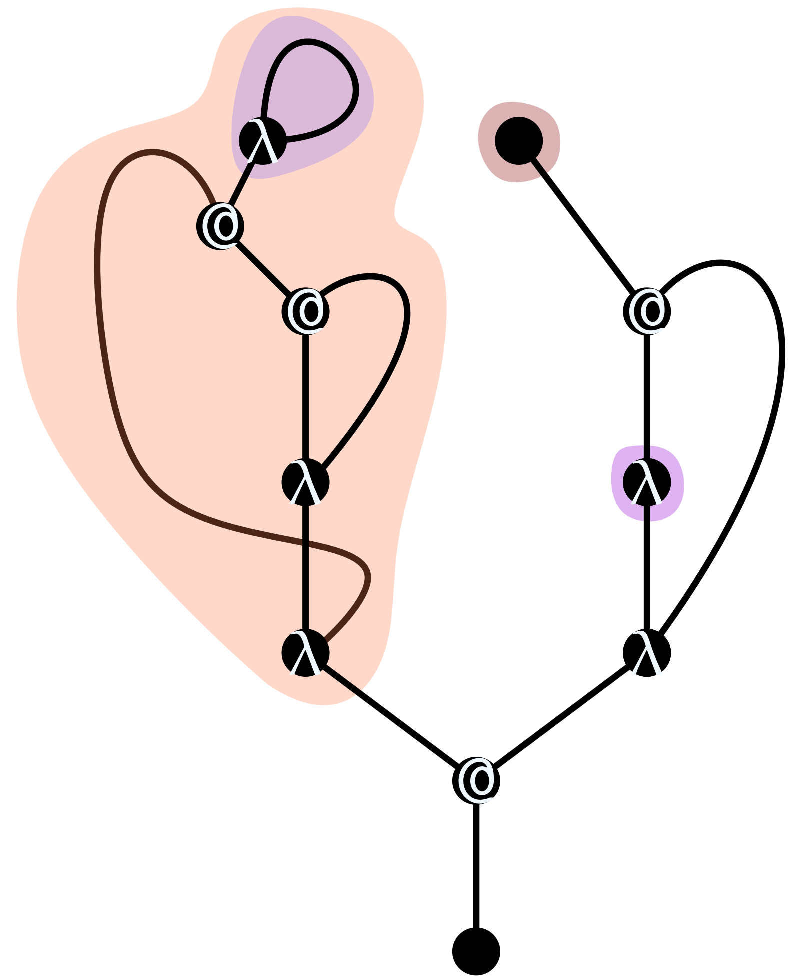
$\bullet = x$



$= \lambda x.t$

order matters!

$= (s\ t)$



Dictionary

- Free var ↔ unary vertex
- Unused λ ↔ binary vertex
- Identity-subterm ↔ loop
- Closed subterm ↔ bridge
- # subterms ↔ # edges

Closed linear terms ↔ trivalent maps
Closed affine terms ↔ (2,3)-valent maps
Established in [BGJ13, BGGJ13]

Why should you, a combinatorialist, be interested in λ-terms?

Decomposing (closed) rooted trivalent maps [BGJ13]

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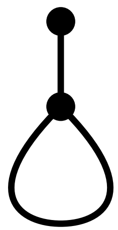

edges

$$\mathsf{T}(z) = z^2$$

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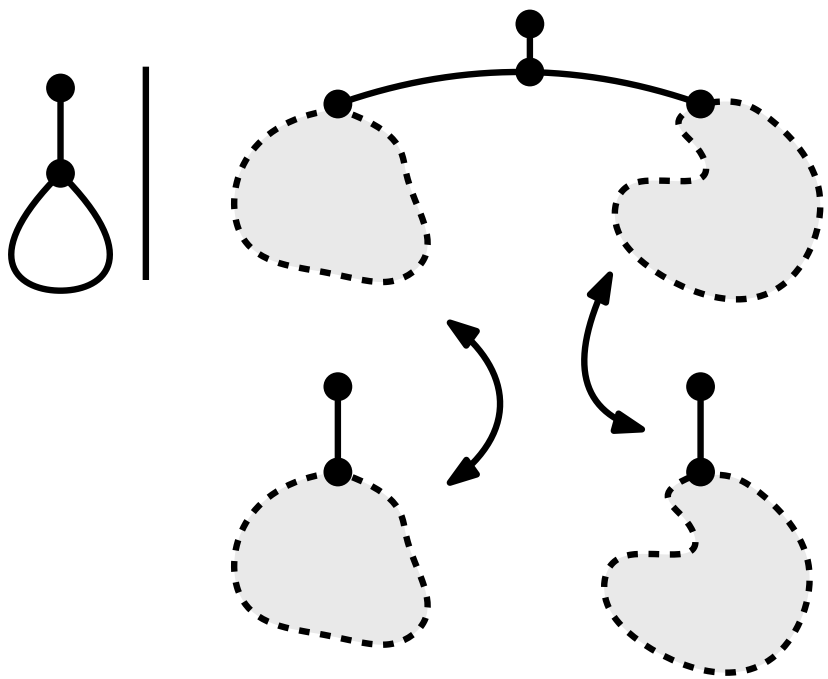

edges

$$\mathsf{T}(z) = z^2 \; + \; z\mathsf{T}(z)^2$$

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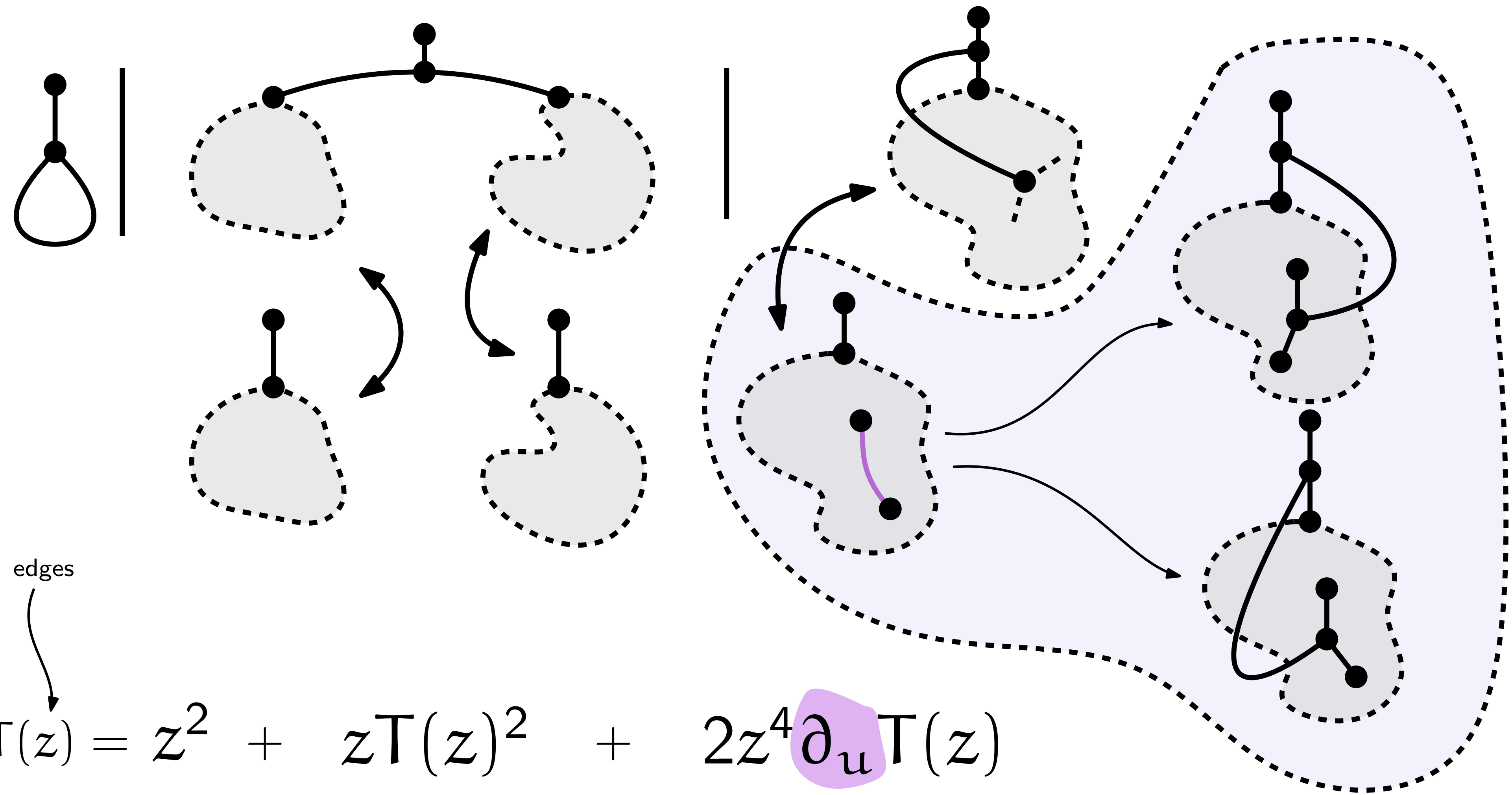
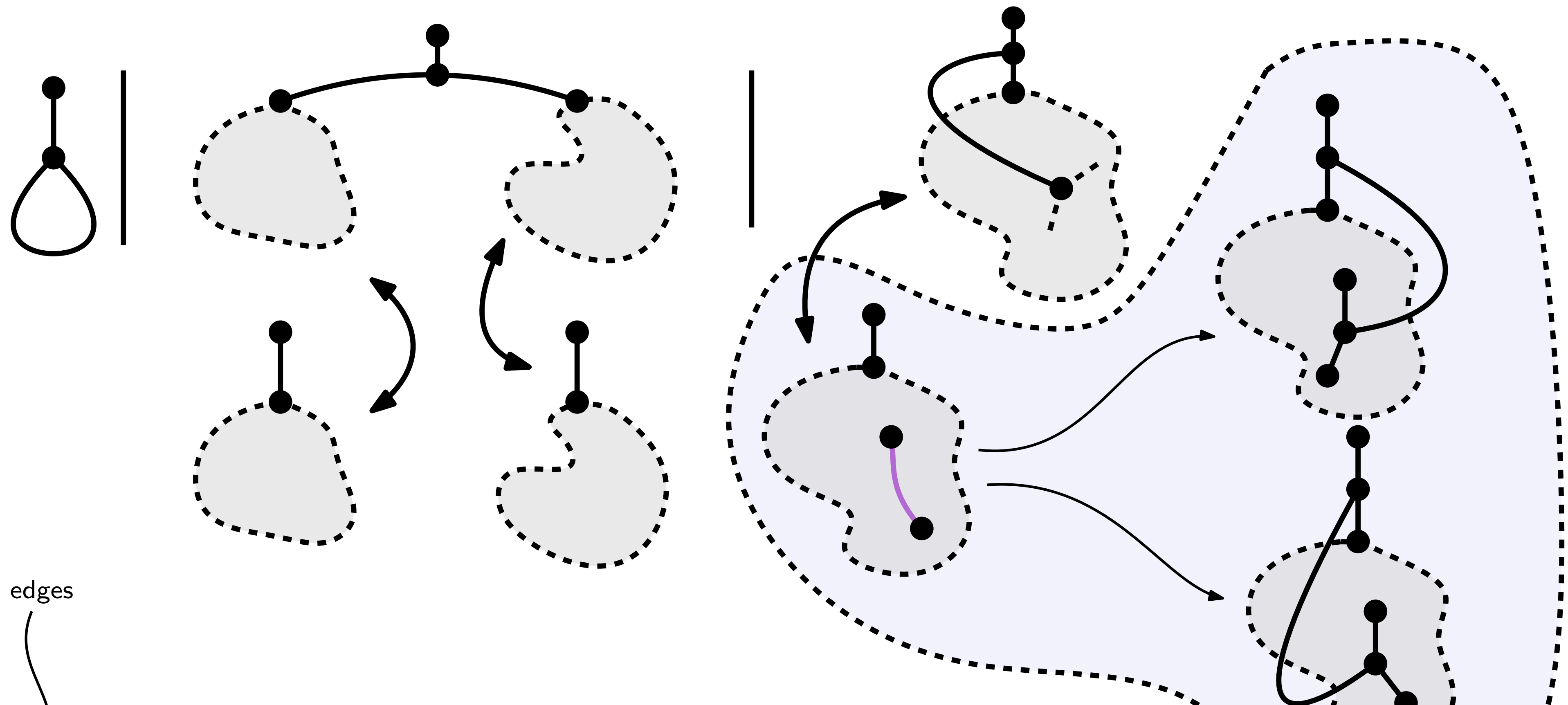


$$\mathsf{T}(z) = z^2 + z\mathsf{T}(z)^2 + 2z^4\partial_u \mathsf{T}(z)$$

Why should you, a combinatorialist, be interested in λ-terms?

Decomposing (closed) rooted trivalent maps [BGJ13]
and closed linear terms!



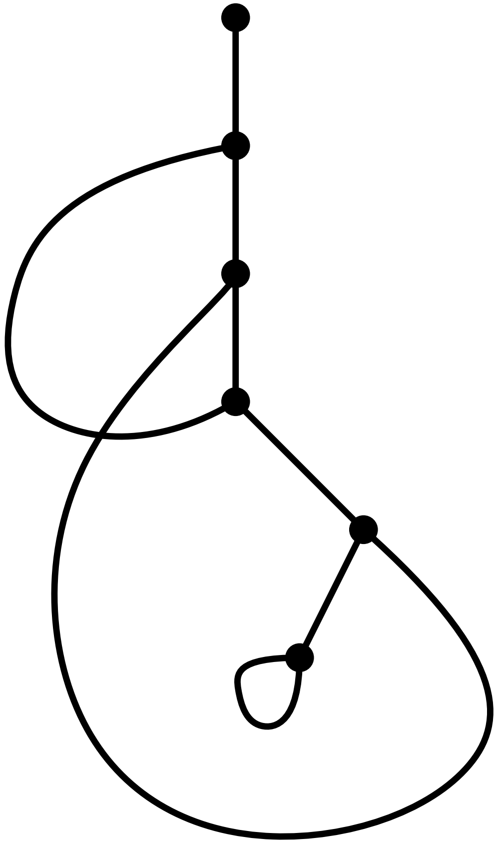
edges

$$\mathsf{T}(z) = z^2 \; + \; z\mathsf{T}(z)^2 \; + \; 2z^4 \partial_u \mathsf{T}(z)$$

subterms

$$\text{lin.term} = \lambda x.x \;\Big|\; (s\ t) \;\Big|\; \begin{array}{l}\lambda x.t[u := (x\ u)] \text{ or} \\ \lambda x.t[u := (u\ x)]\end{array}$$

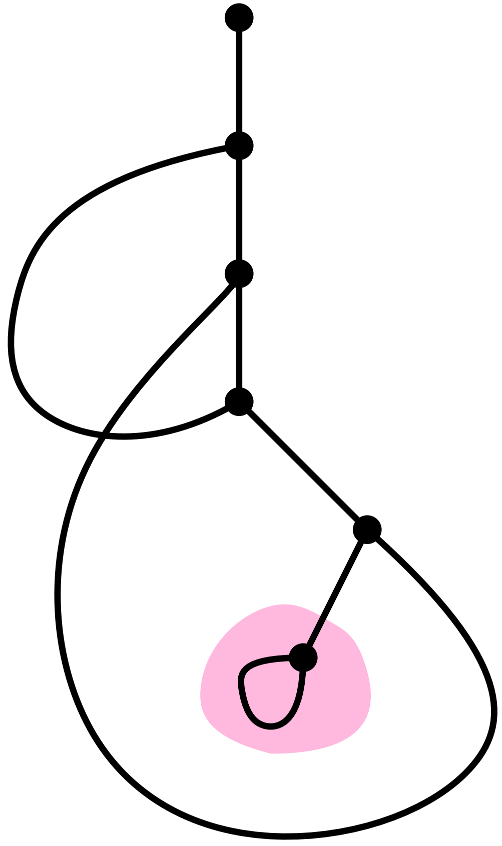
Some of our previous results: limit distributions



$$\lambda x.\lambda y.(y\ \lambda w.w)x$$

Some of our previous results: limit distributions

\# loops = \# id-subterms



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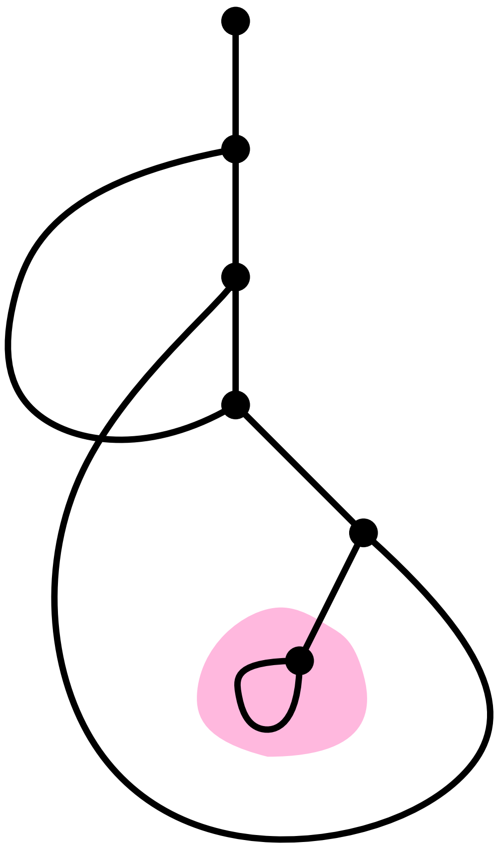

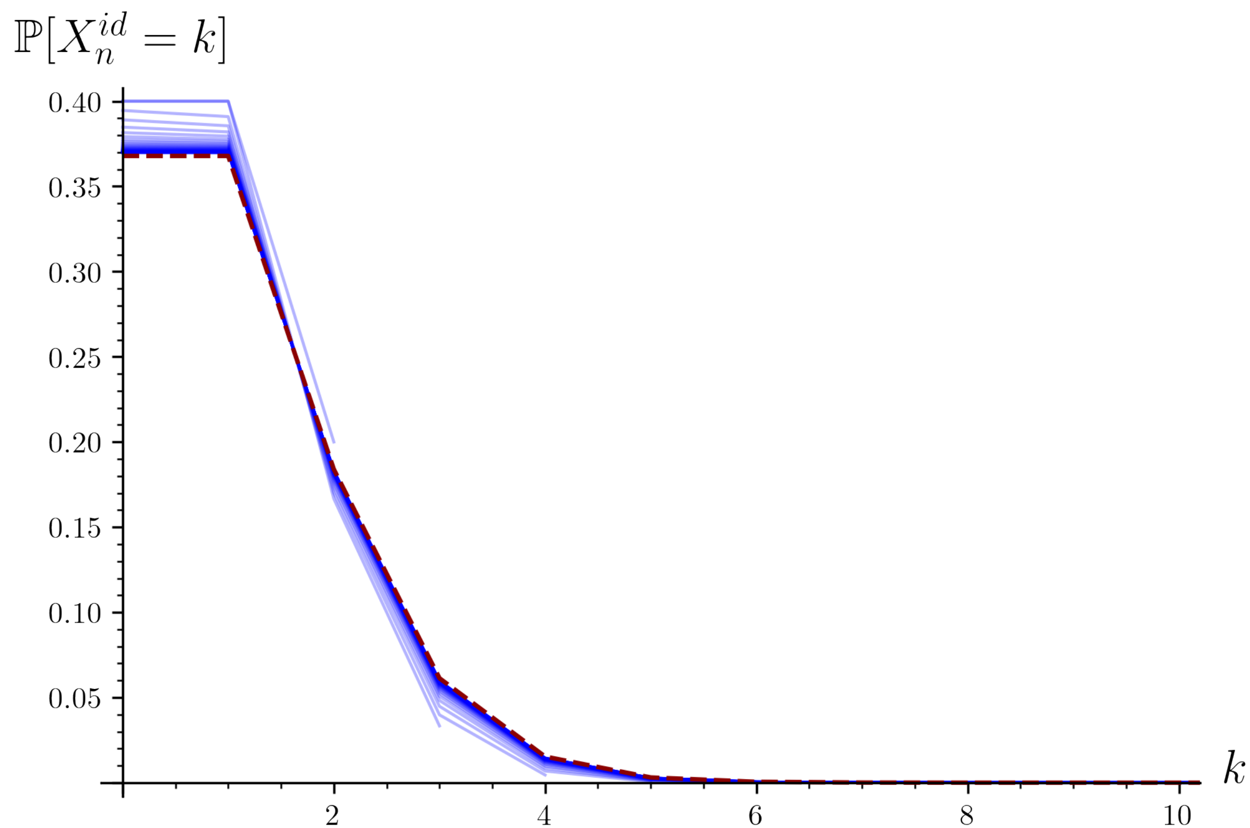

$$\lambda x.\lambda y.(y\ \lambda w.w)x$$

$$X_n^{\mathrm{id}} \xrightarrow{D} \mathrm{Poisson}(1)$$

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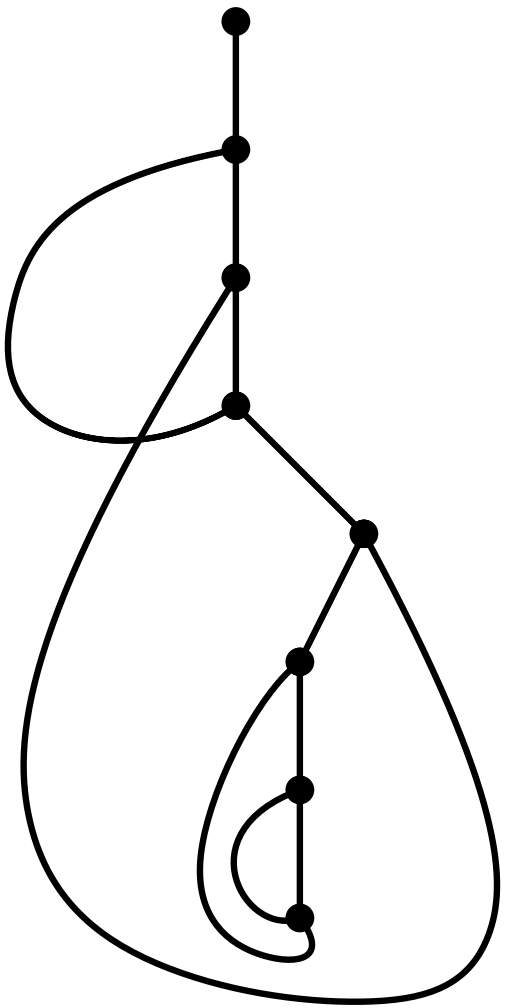
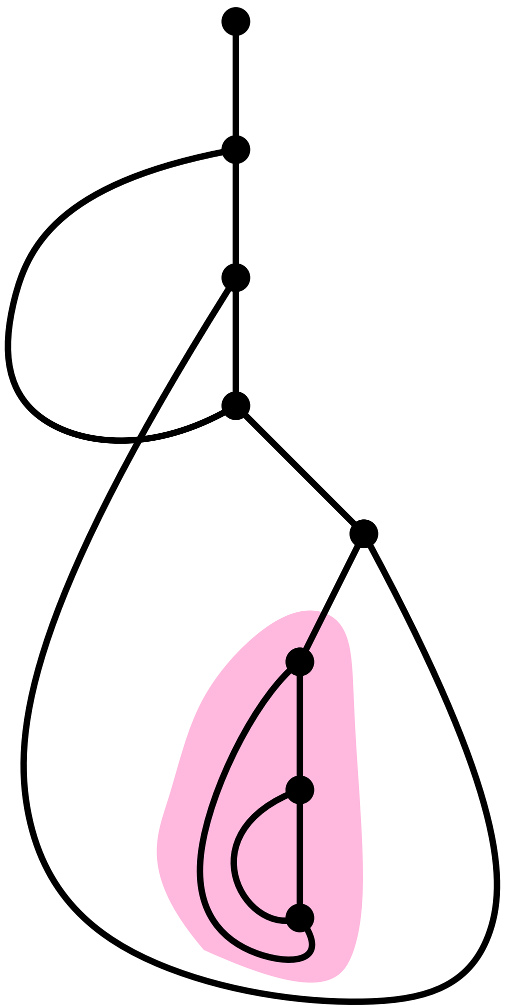


$\lambda x.\lambda y.(y\ \lambda z.\lambda w.zw)x$

Some of our previous results: limit distributions

$\#$ bridges $= \#$ closed subterms



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Some of our previous results: limit distributions

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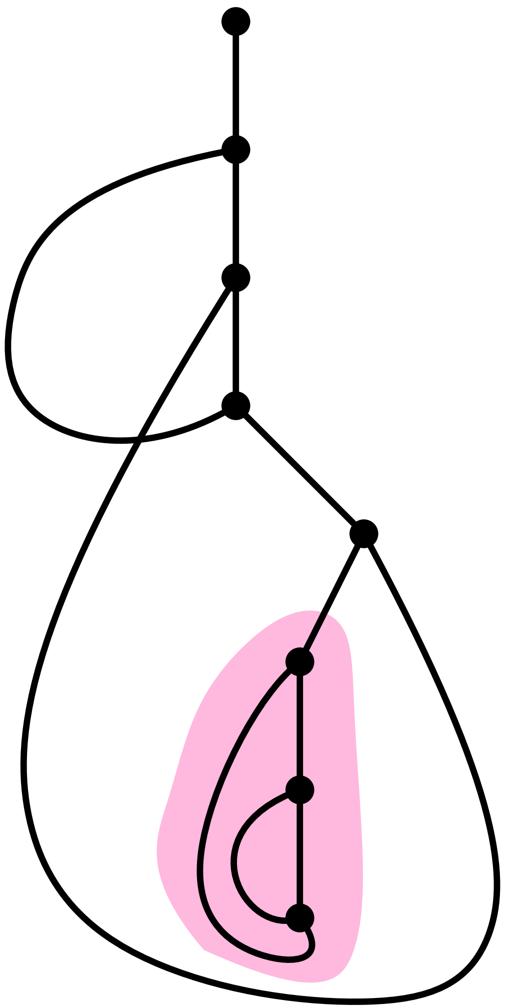

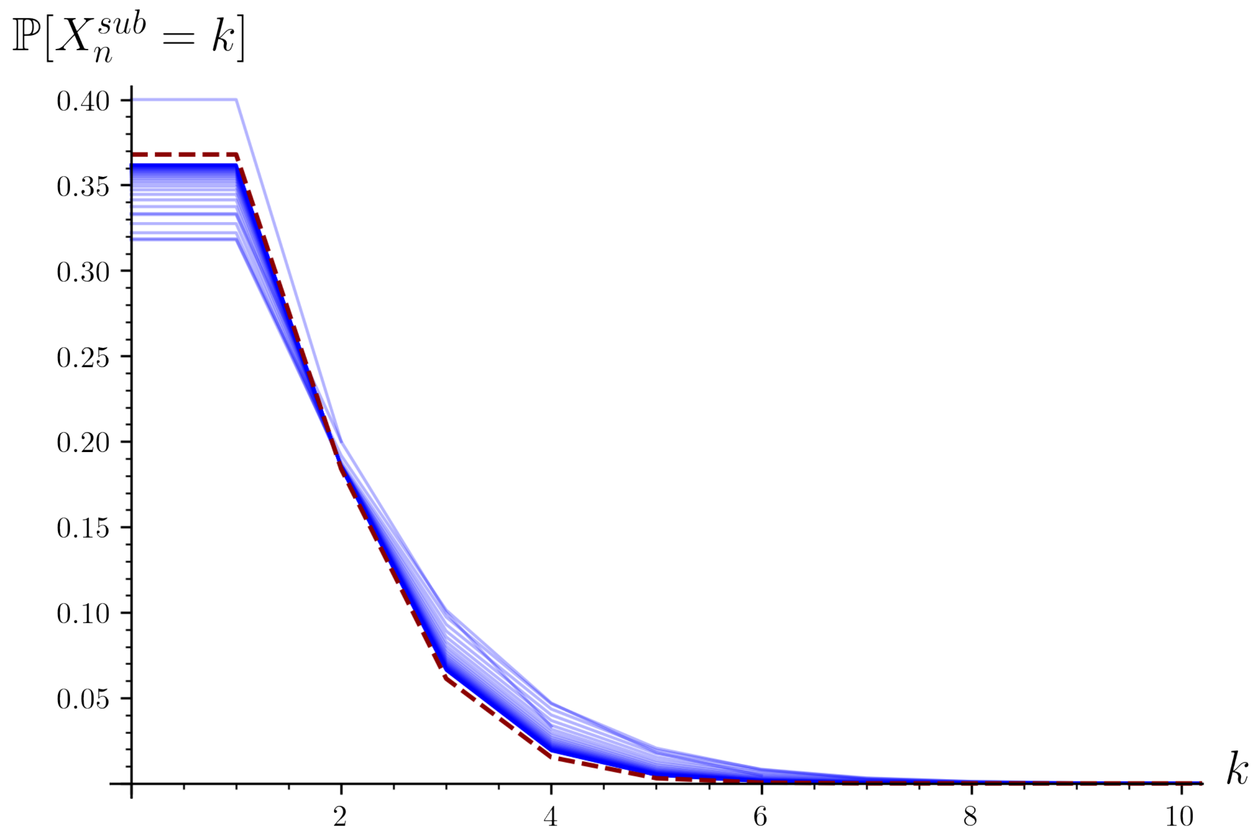

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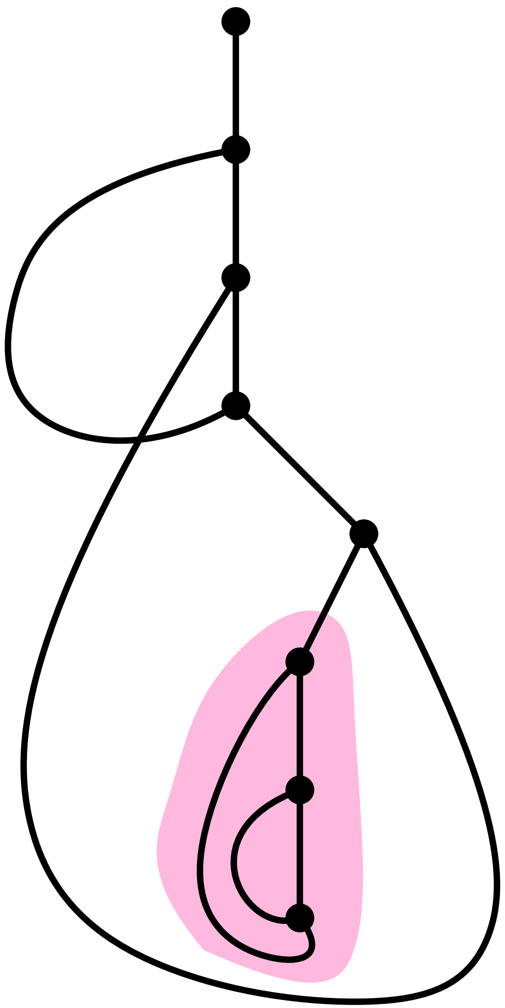

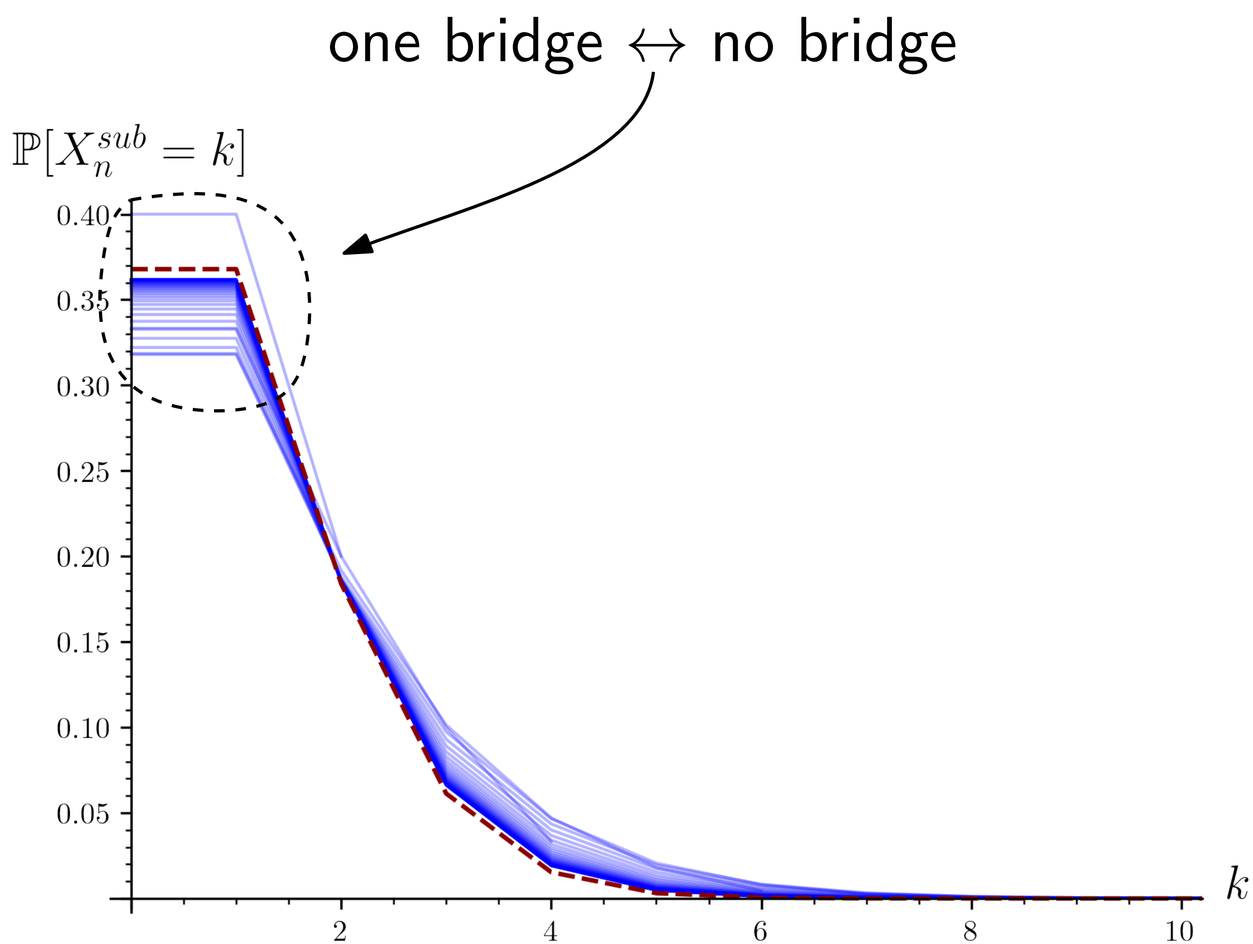

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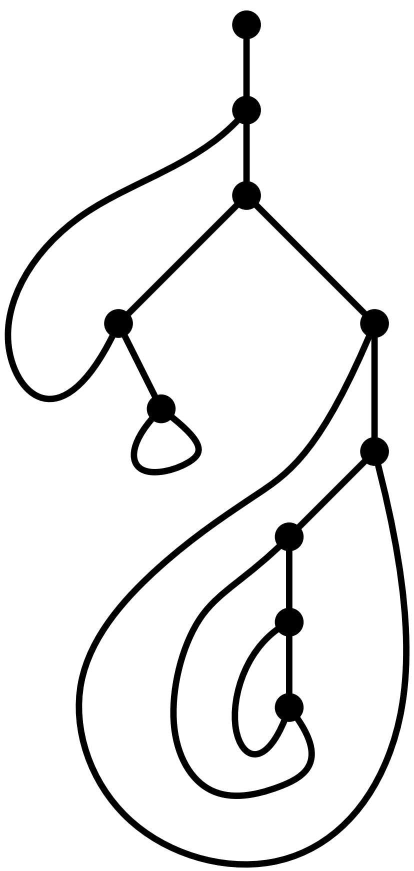
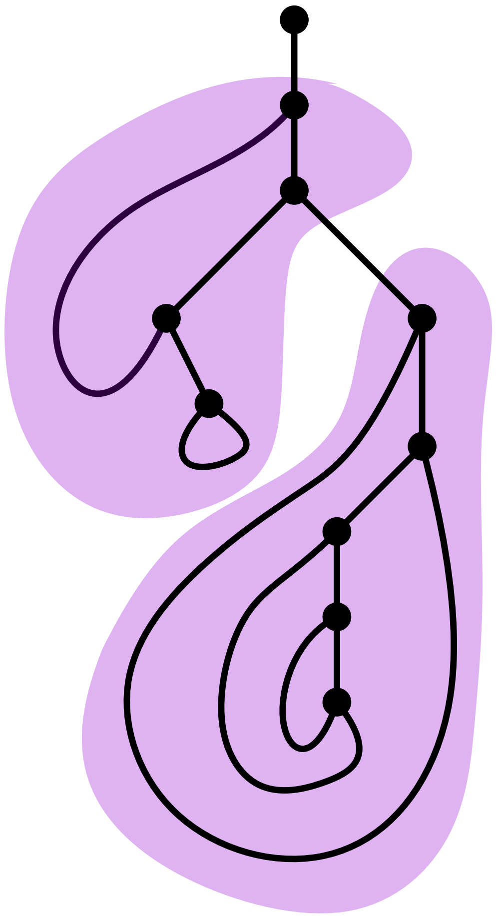
β-reduction as map rewriting



$\lambda x.(\lambda y.y(\lambda z.\lambda w.zw))((\lambda u.u)x)$

β-reduction as map rewriting



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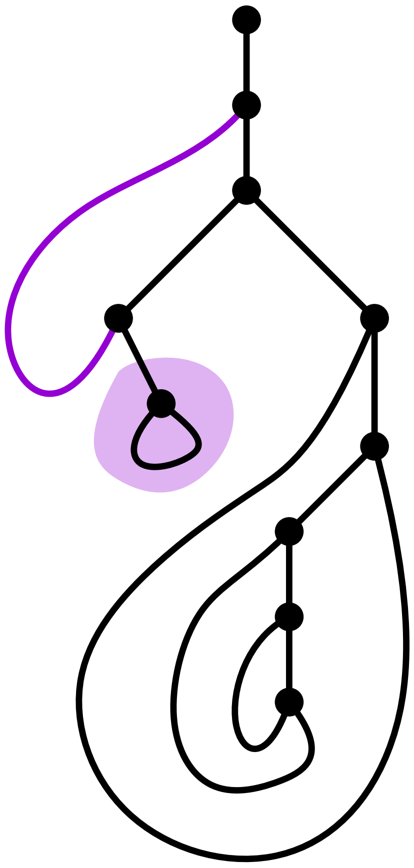

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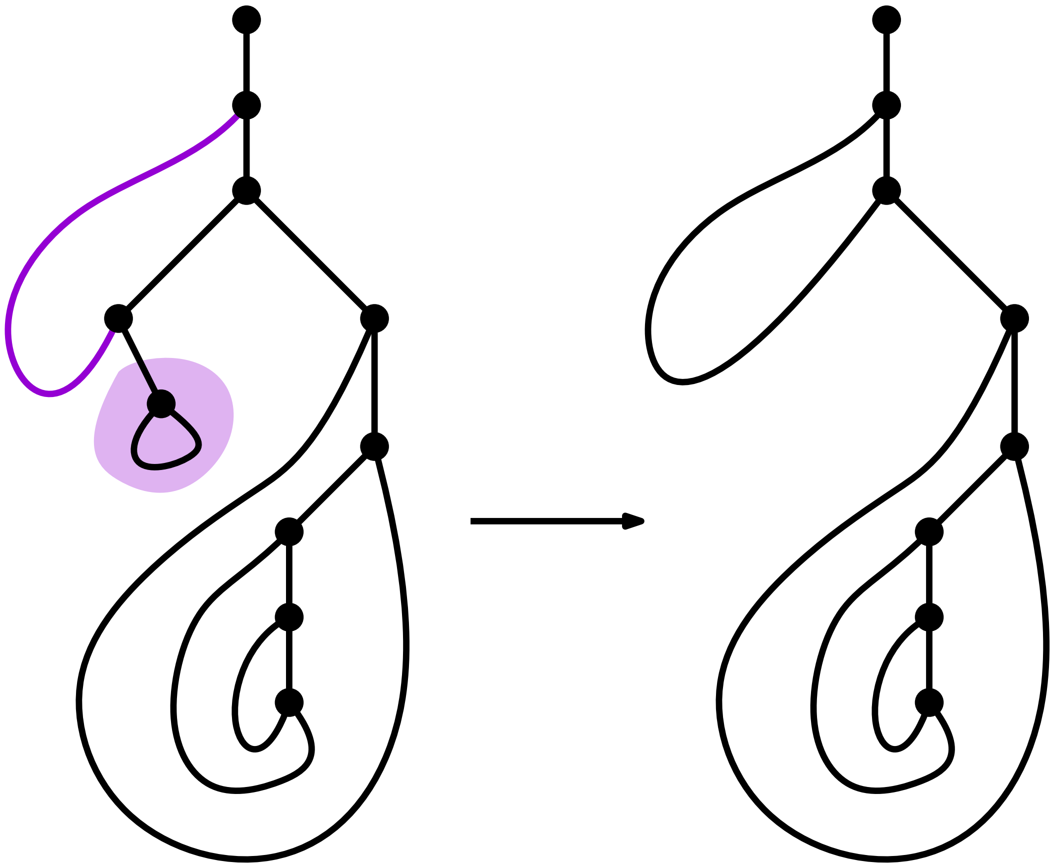
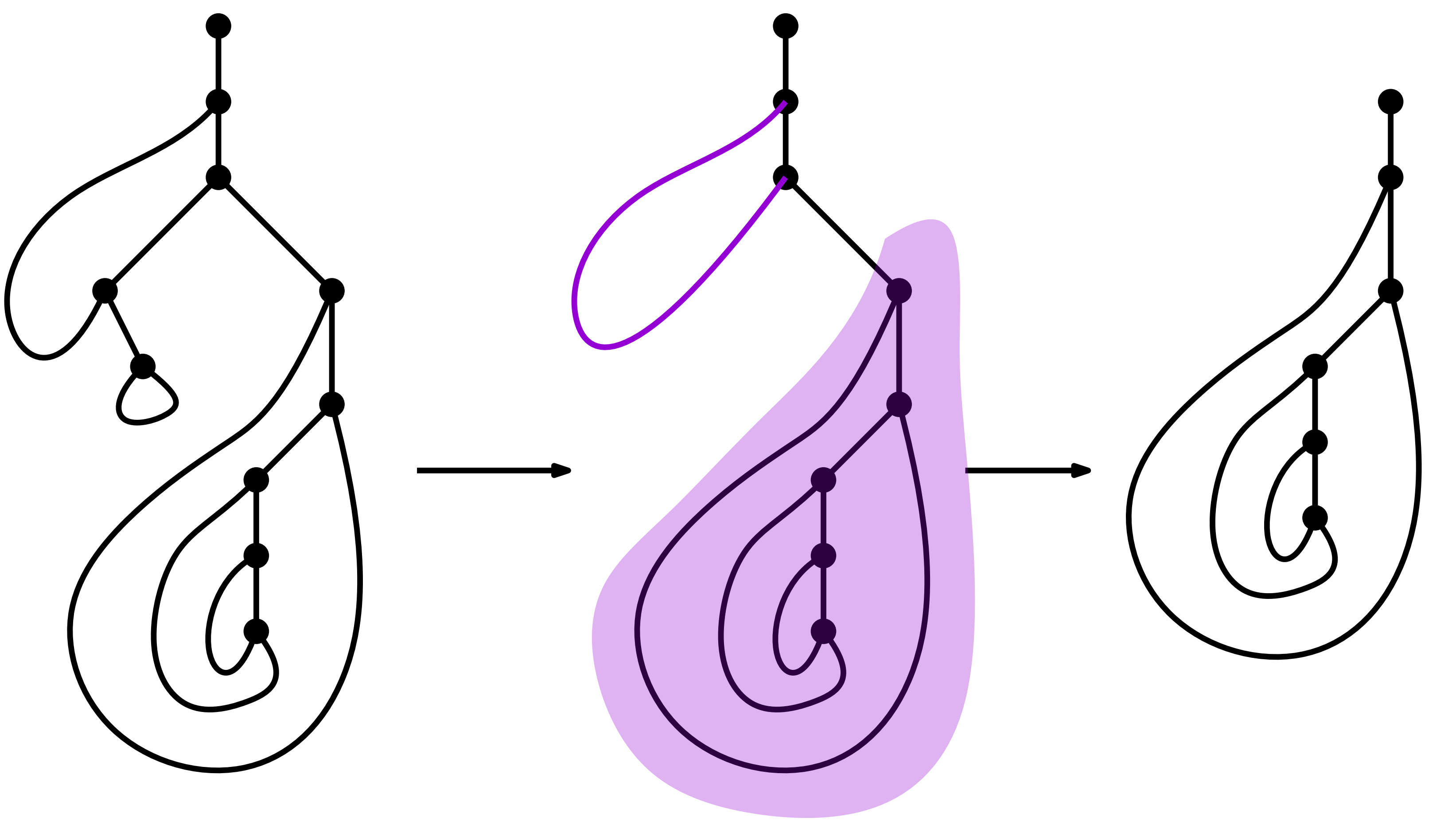


$$\lambda x.(\lambda y.y(\lambda z.\lambda w.zw))((\lambda u.u)x)$$

$$\longrightarrow \lambda x.((\lambda y.y(\lambda z.\lambda w.zw))x)$$

β-reduction as map rewriting



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$\lambda x.x\ (\lambda z.\lambda w.zw)$

Our workflow:

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first order asymptotics given by $k = n_0$ and $k = n$

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crucial ingredient: coefficients are growing *rapidly*

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Mean number of β-redices in closed terms

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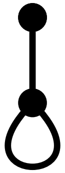
no redex



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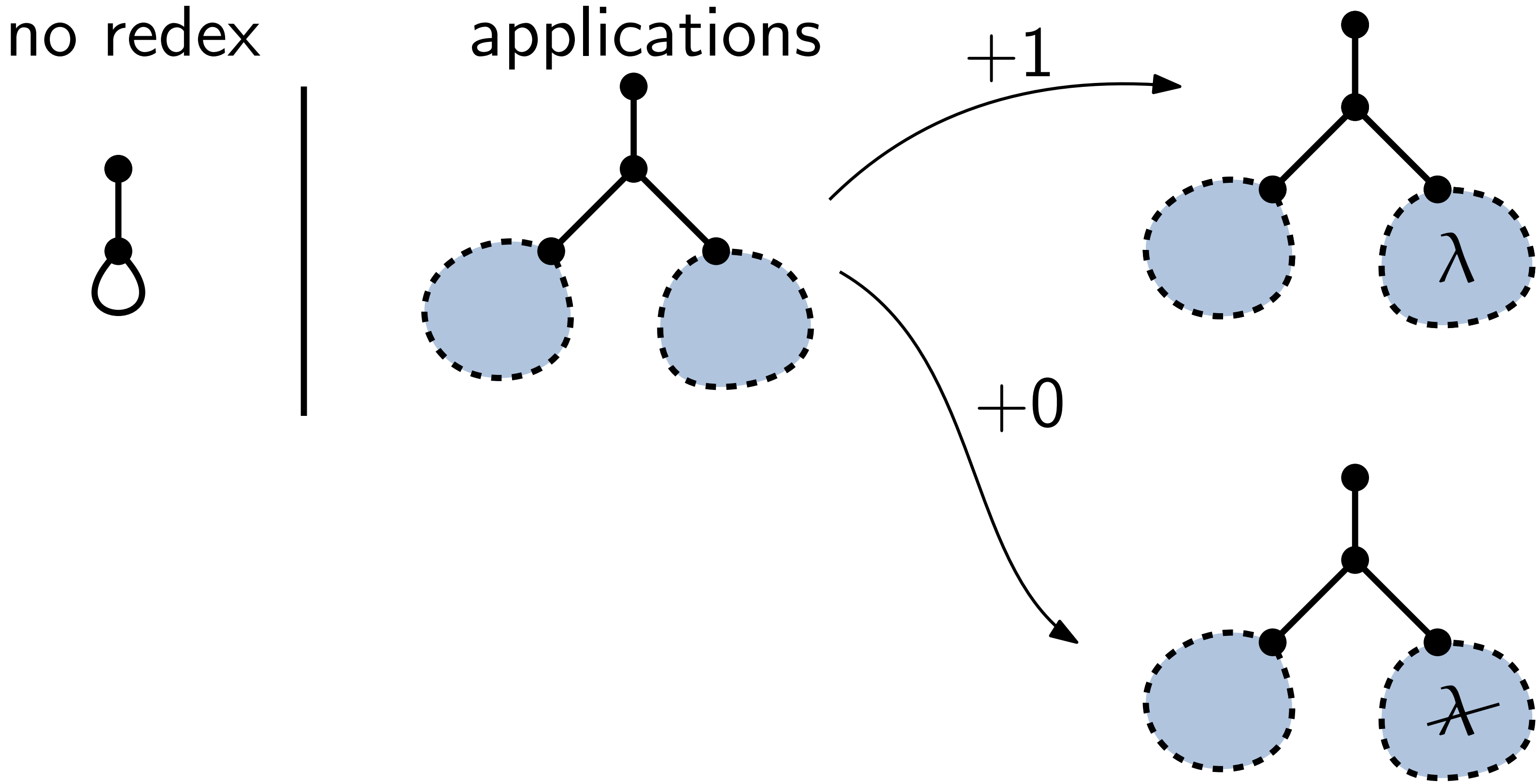
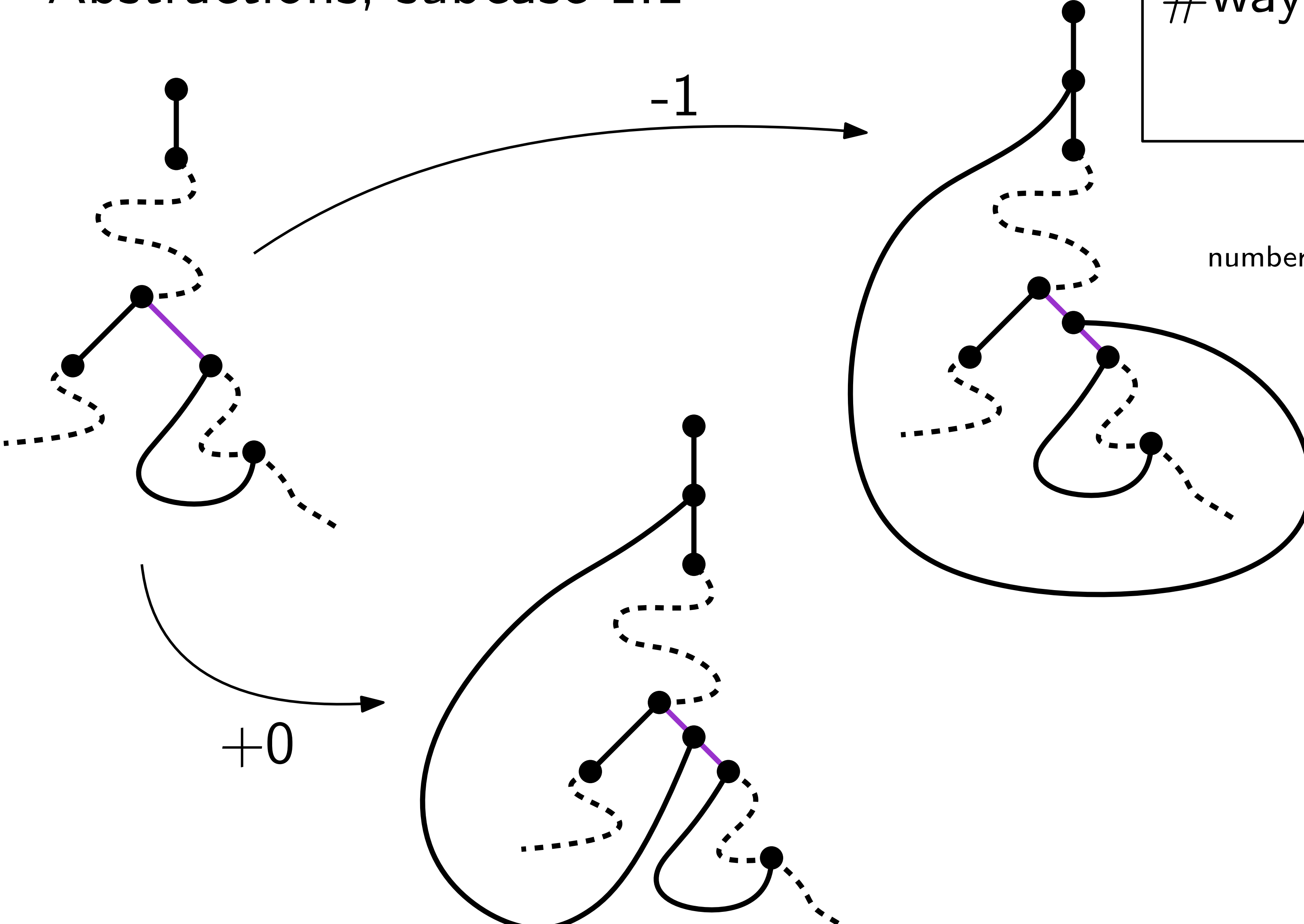


Mean number of β-redices in closed terms

- Tracking redices during the decomposition

Abstractions, subcase 1.1



$\#$ways to do this

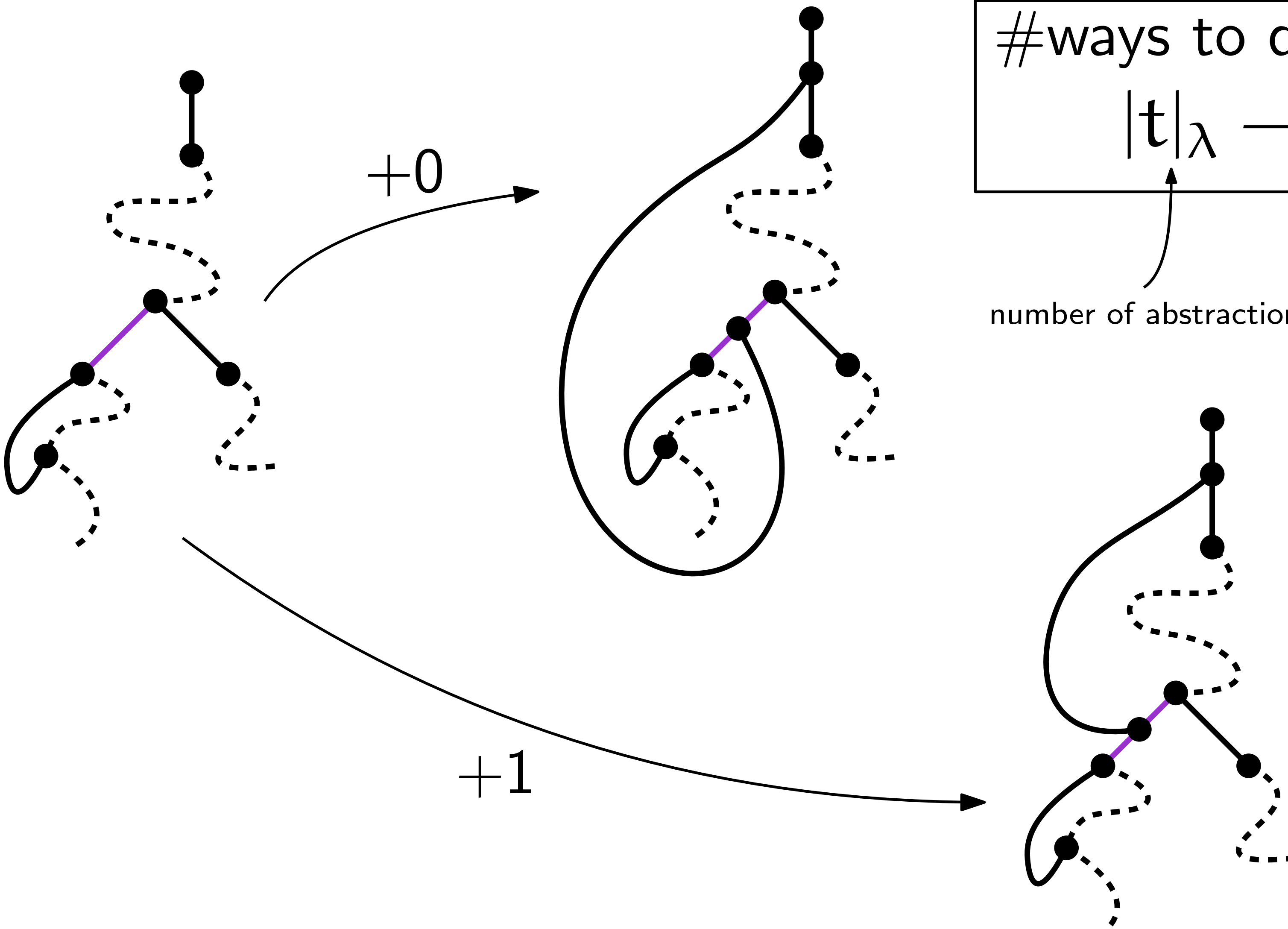
$$|t|_\beta$$

number of redices in t

Mean number of β-redices in closed terms

- Tracking redices during the decomposition

Abstractions, subcase 1.2



#ways to do this

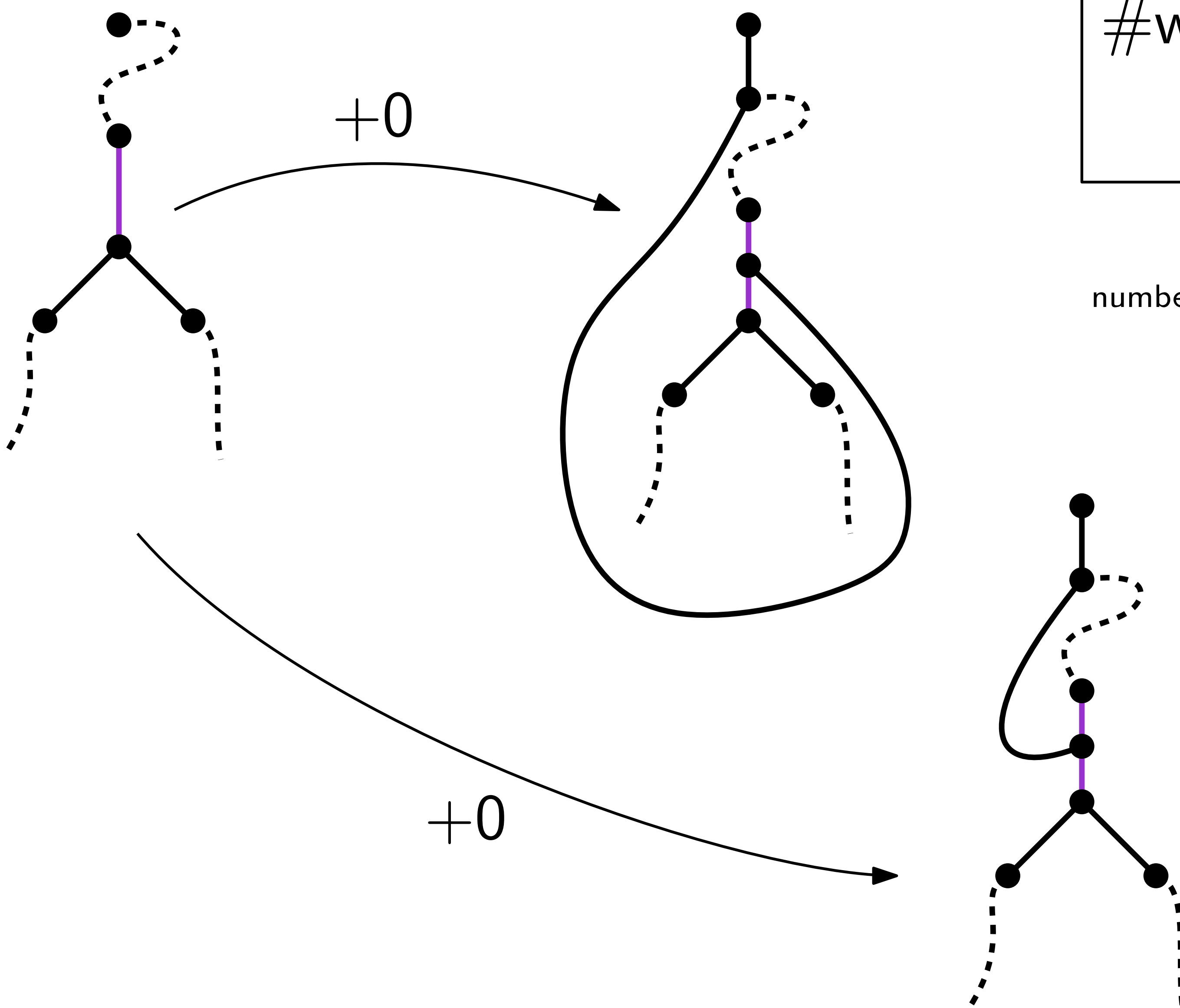
$$|t|_\lambda - |t|_\beta$$

number of abstractions in t

Mean number of β-redices in closed terms

- Tracking redices during the decomposition

Abstractions, subcase 1.3



#ways to do this

$$|t| - |t|_\lambda$$

number of subterms in t = size of t

Mean number of β-redices in closed terms

- Building the specification of the OGF

- $|t|_\lambda = \frac{|t|+1}{3}, |t| - |t|_\lambda = \frac{2|t|-1}{3}$
- $r\partial_r T_0 = \sum_{t\in T_0} |t|_\beta z^{|t|} r^{|t|_\beta}$
- $\frac{z\partial_z T_0 + T_0}{3} = \sum_{t\in T_0} \frac{|t|+1}{3} z^{|t|} v^{|t|_\beta}$
- $\frac{2z\partial_z T_0 - T_0}{3} = \sum_{t\in T_0} \frac{2|t|-1}{3} z^{|t|} v^{|t|_\beta}$

Mean number of β-redices in closed terms

- Translating to a differential equation and pumping

$$T_0 = -z\left(z^2(r+1)(1+(r-1)zT)(r-1)\partial_r T_0 - \frac{(1+z(r-1)T)z^3(r+5)\partial_z T_0}{3} - \frac{z^3(r-1)^2 T_0^2}{3} - \frac{4z^2(r-1)T_0}{3} - z - T_0^2\right)$$

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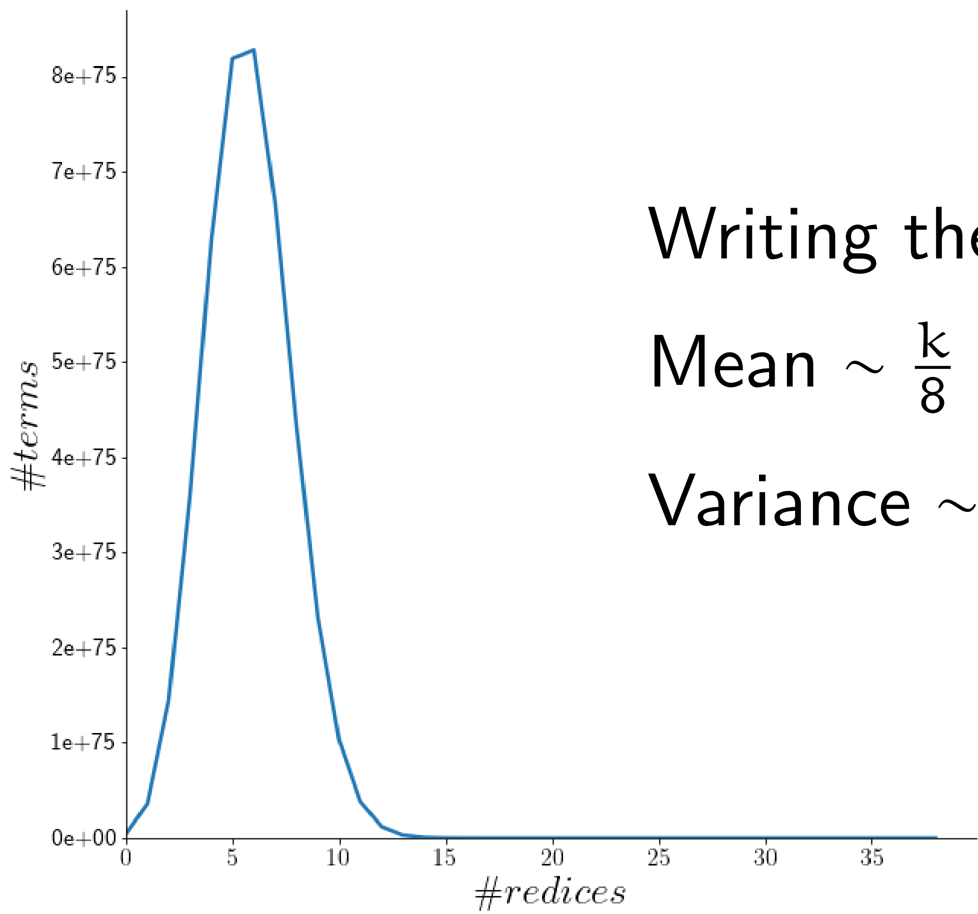
A plot of the dist. of redices for terms/maps of size $n = 119$



Writing the size as $n = 3k + 2$, we have:

Mean $\sim \frac{k}{8}$

Variance $\sim \frac{29k}{320}$

Whats next?

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- Tracking the three patterns whose reduction alters the number of redices (WIP with Bodini, Zeilberger, Wallner, Gittenberger)

$$(\lambda x.C[(x\ u)])(\lambda y.t_2) \qquad ((\lambda x.\lambda y.t_1)t_2)t_3$$

$$(\lambda x.x)(\lambda y.t_1)t_2$$

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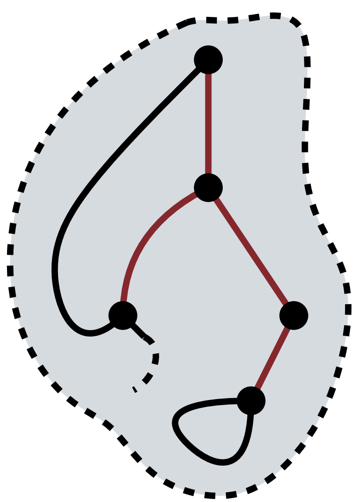
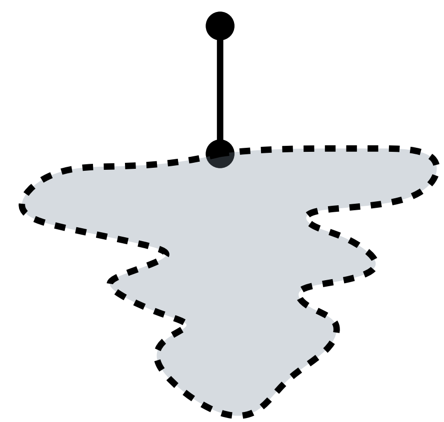
- More parameters:



Mean path length



Profile

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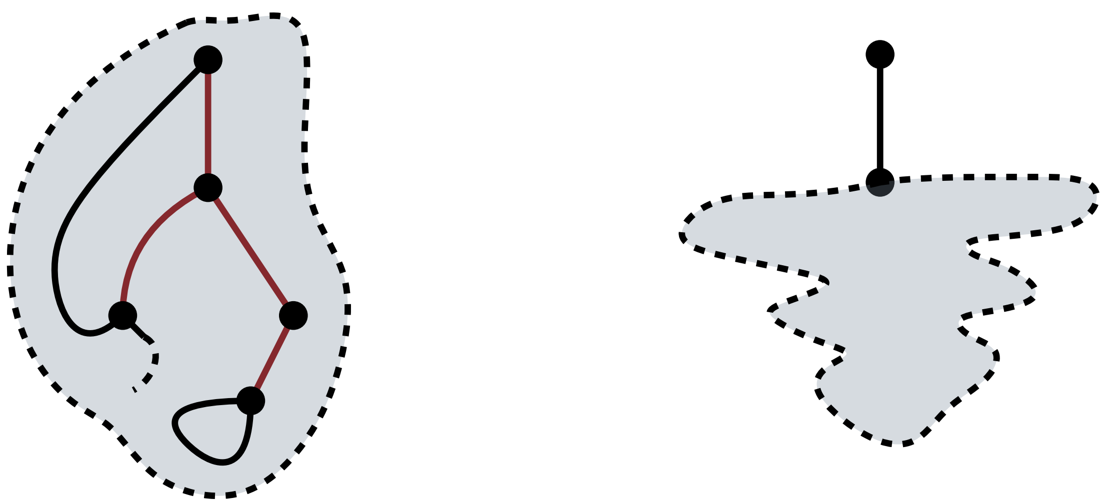

Mean path length   Profile

Thank you!

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