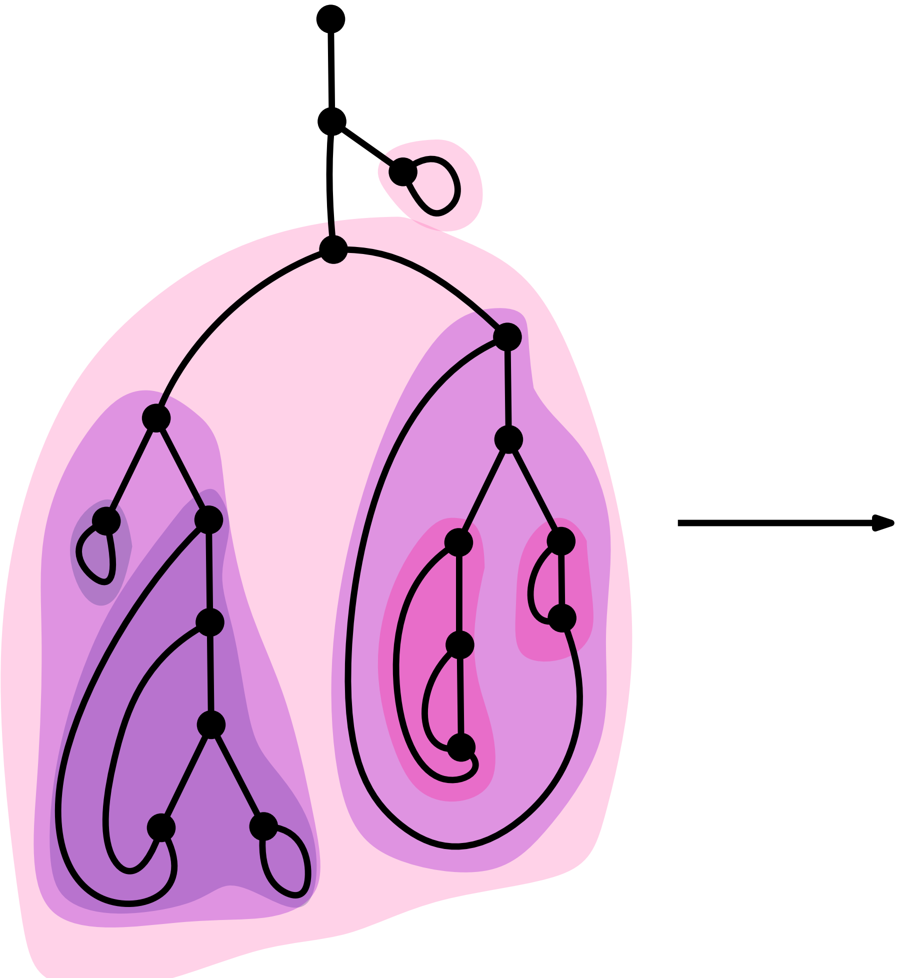
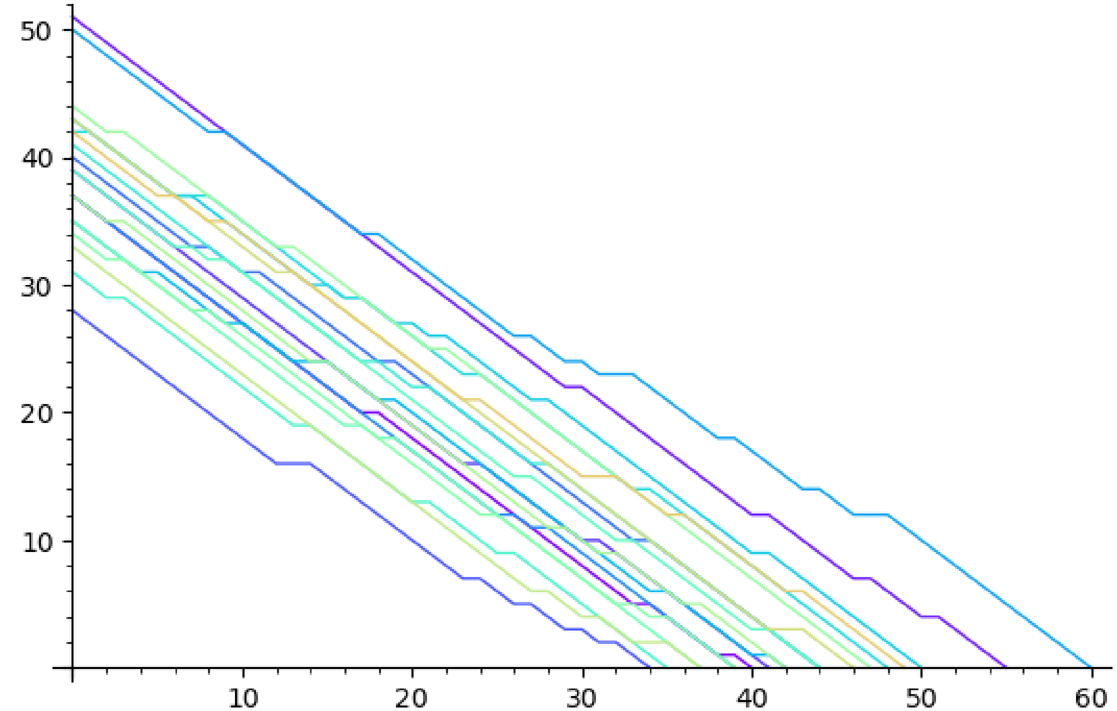
A lower bound on reduction length for random closed linear λ-terms





Algorithmic and Enumerative Combinatorics, 7 July 2022

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Bernhard Gittenberger (TU WIen)
Noam Zeilberger (LIX, Polytechnique)
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The linear λ-calculus

- A **PTIME-complete** system of computation [M04]

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variables

abstractions
represent functions "$x \mapsto t$"
x appears **exactly once** inside t

applications
represent "$f(t)$"

- Terms considered up to (careful) renaming of variables:

$$(\lambda x.\lambda y.(x\ y)) = (\lambda x.\lambda z.(x\ z)) \neq (\lambda x.\lambda y.(x\ a))$$

Examples of linear λ-terms

$(\lambda x.(x\ y))$ open term

$(\lambda x.x)$ closed term

$(y\ (\lambda z.z))$ open term with closed subterm

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Dynamics of the λ-calculus: β-reductions

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represents:

$$f = x \mapsto t_1$$

$f(t_2)$: replace x with t_2 inside t_1

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A term with no redices is called a normal form

- Repeated β-reduction terminates with a unique normal form
- Starting from a random term, how many steps to reach the normal form?

A *lower bound* is given by the number of β-redices!

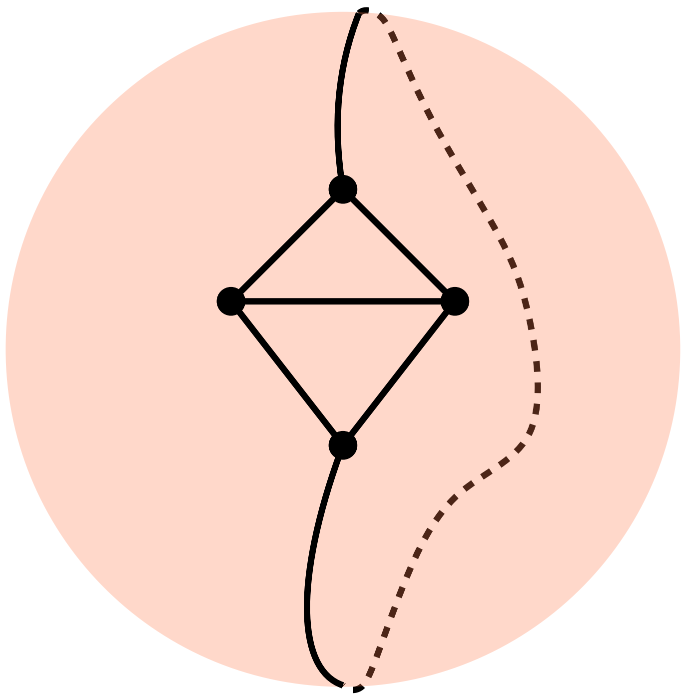
This motivates our first problem-to-solve:

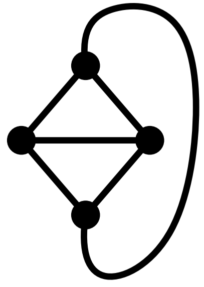
What is the number of β-redices in a random linear λ-term?

random variable (number of β-redices)

uniform distribution (random linear λ-term)

What are maps?





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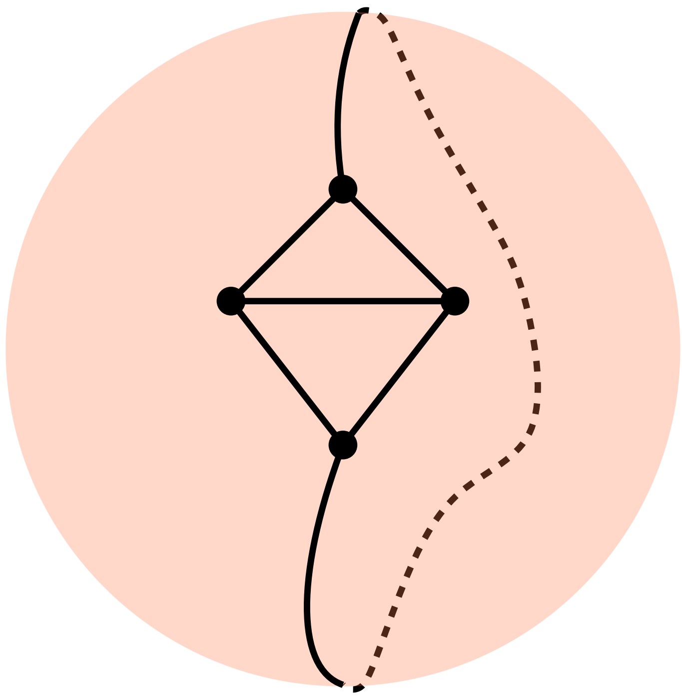

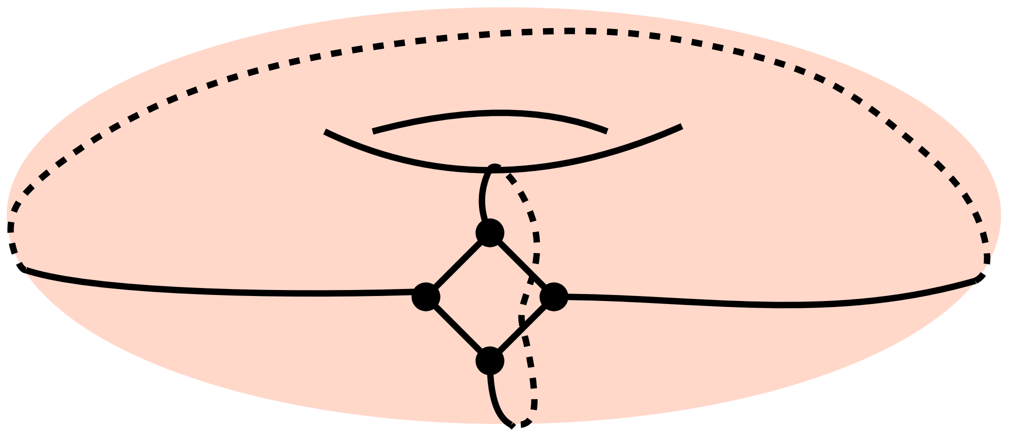

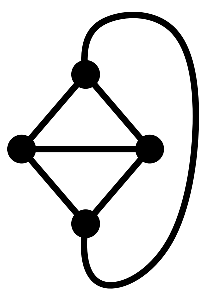

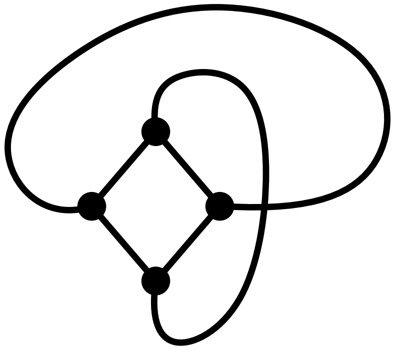

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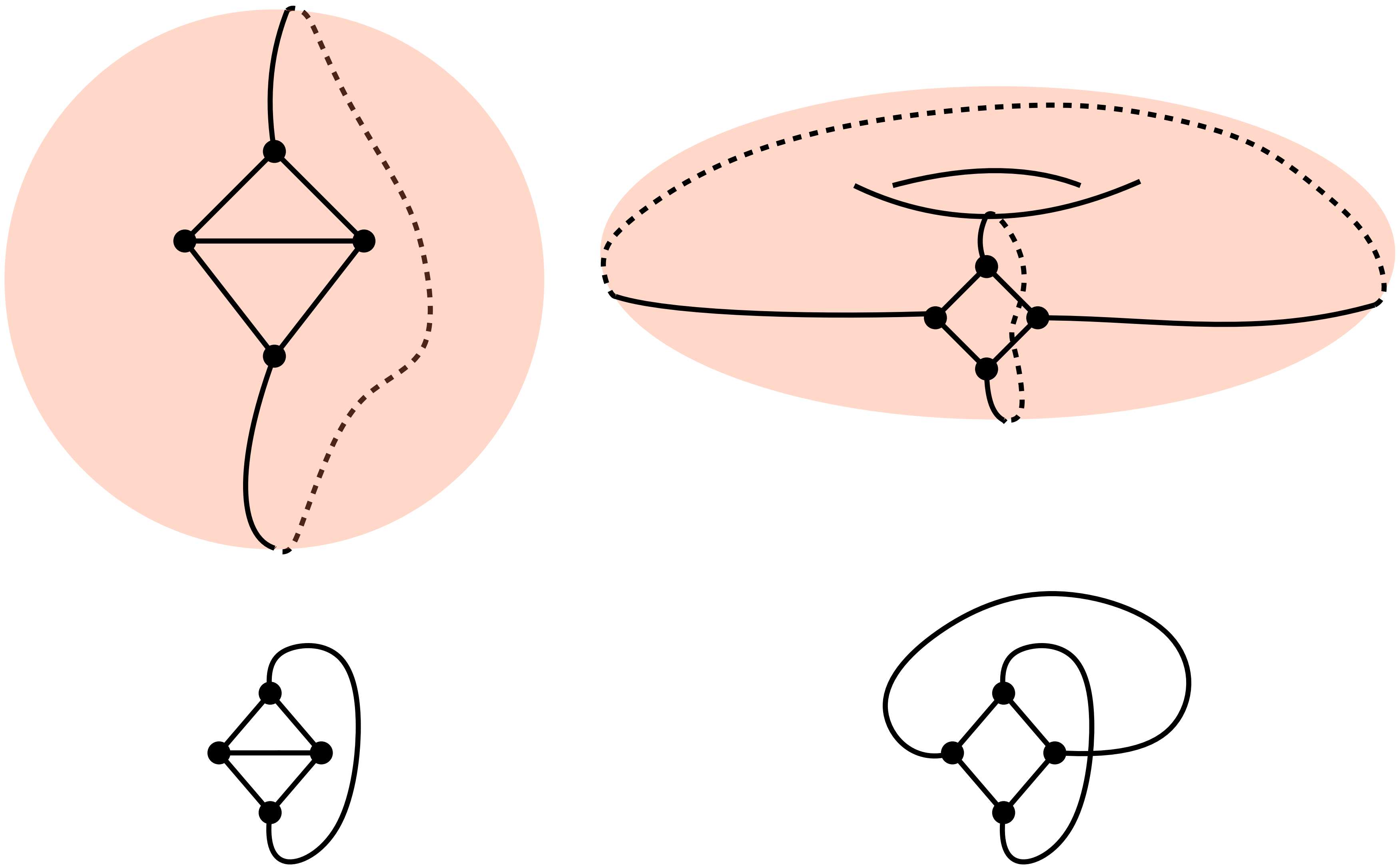
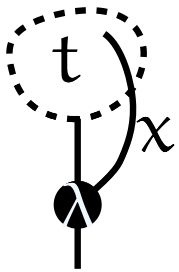


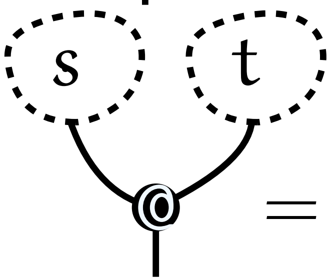
We're interested in unrestricted genus cubic maps

Why should logicians be interested in maps?

$= x$



$= \lambda x.t$



$= (s\ t)$

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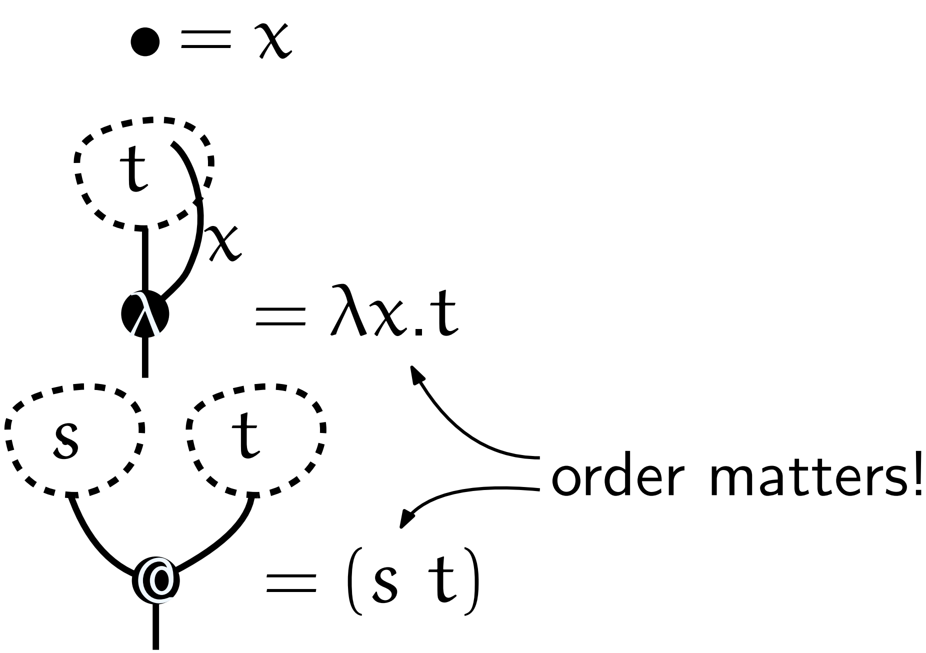

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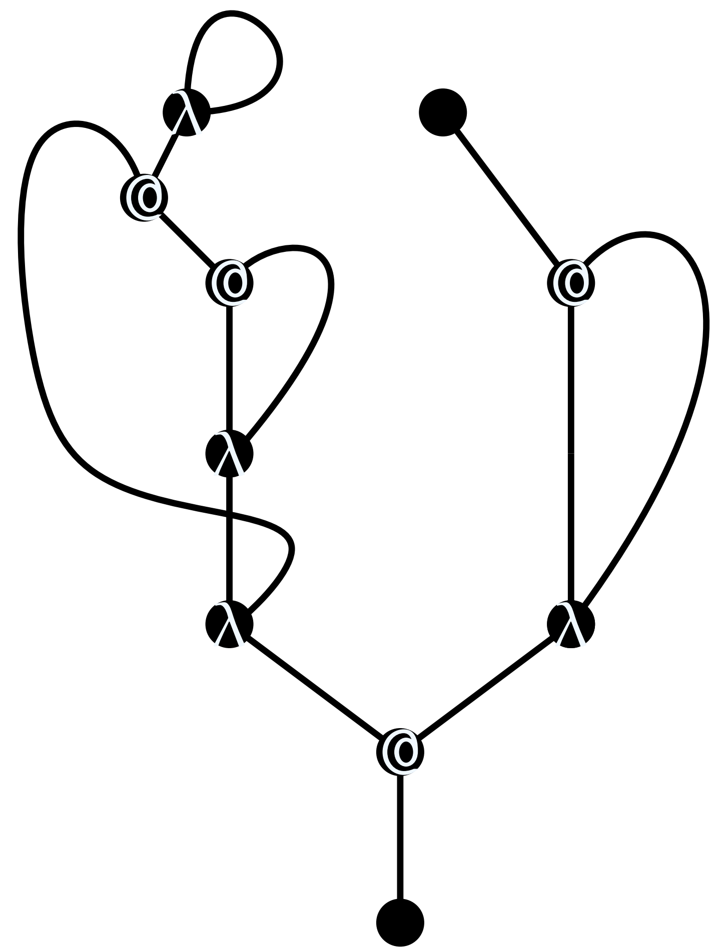

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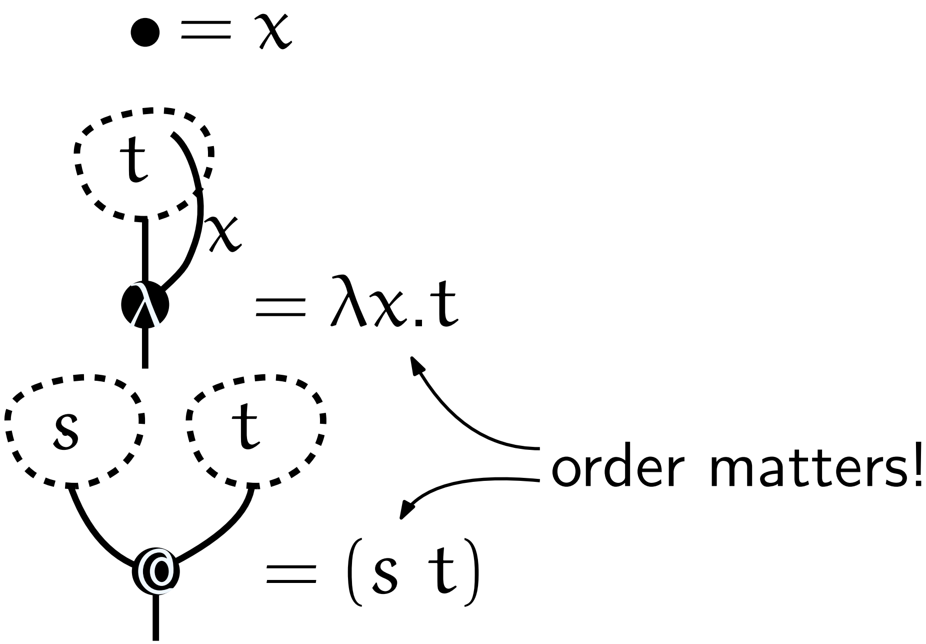

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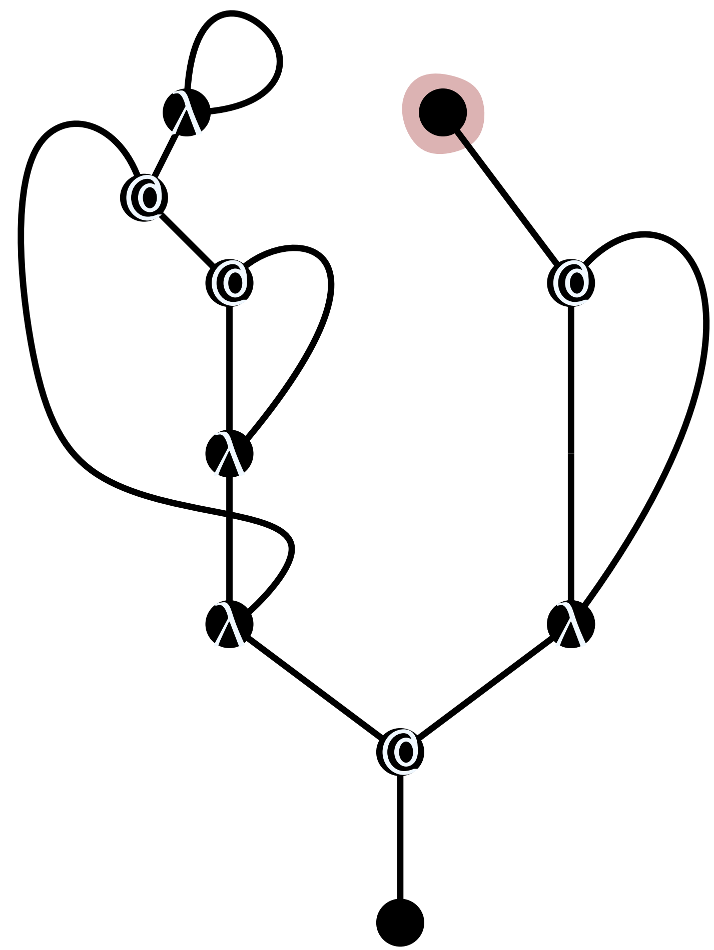

Dictionary

- Free var ↔ unary vertex

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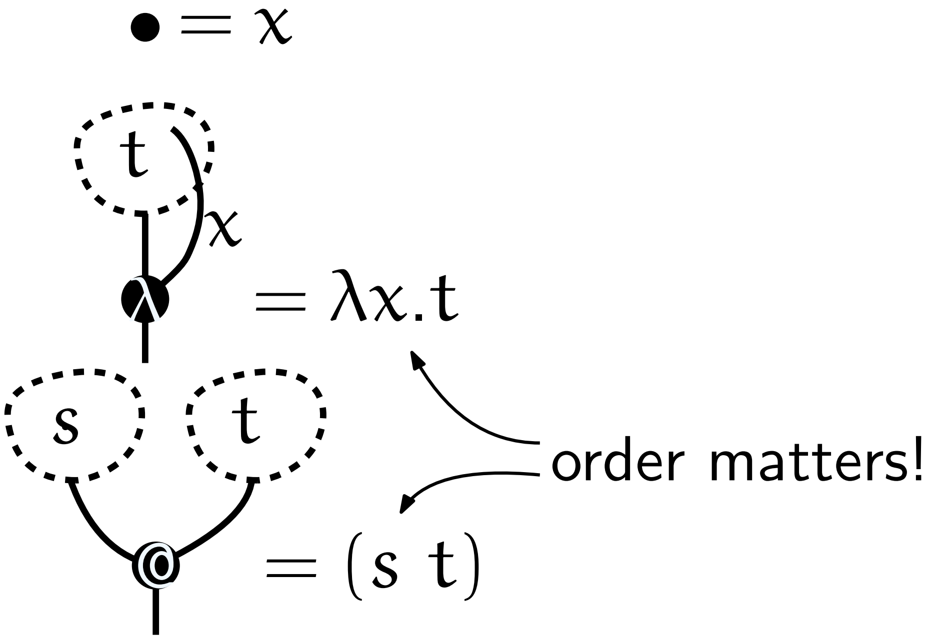

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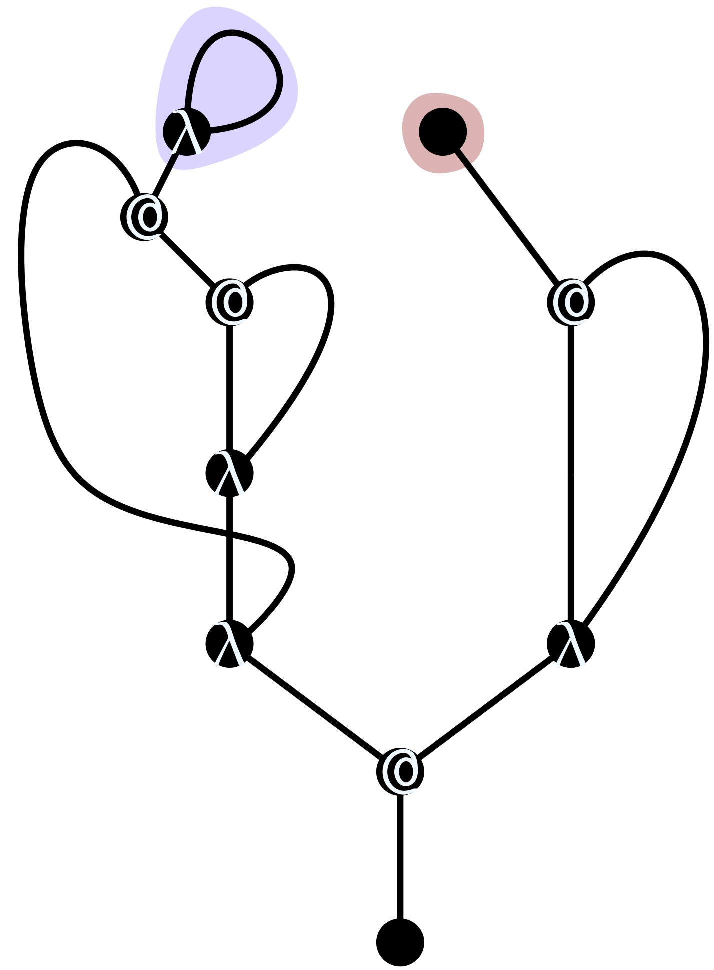

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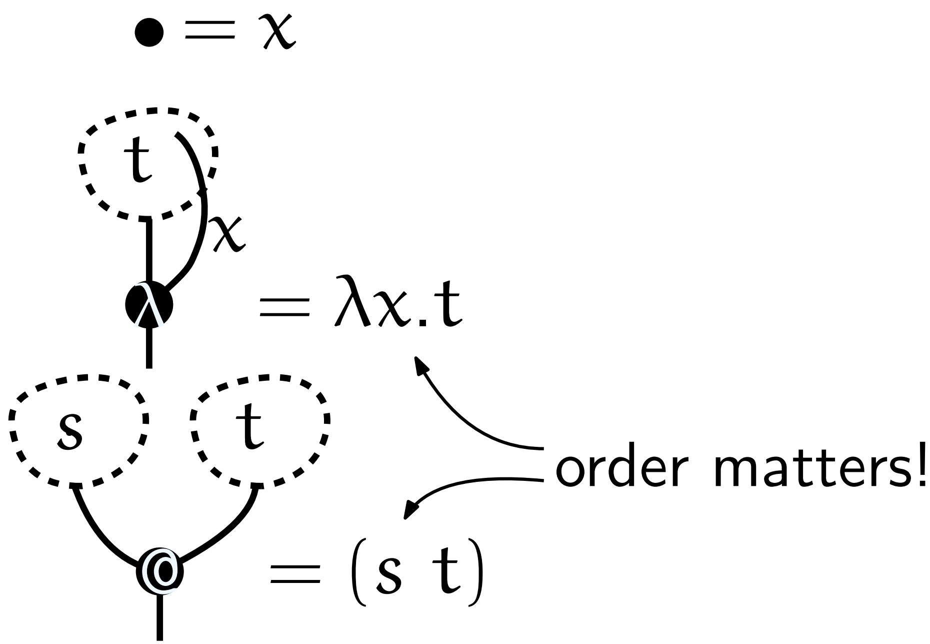

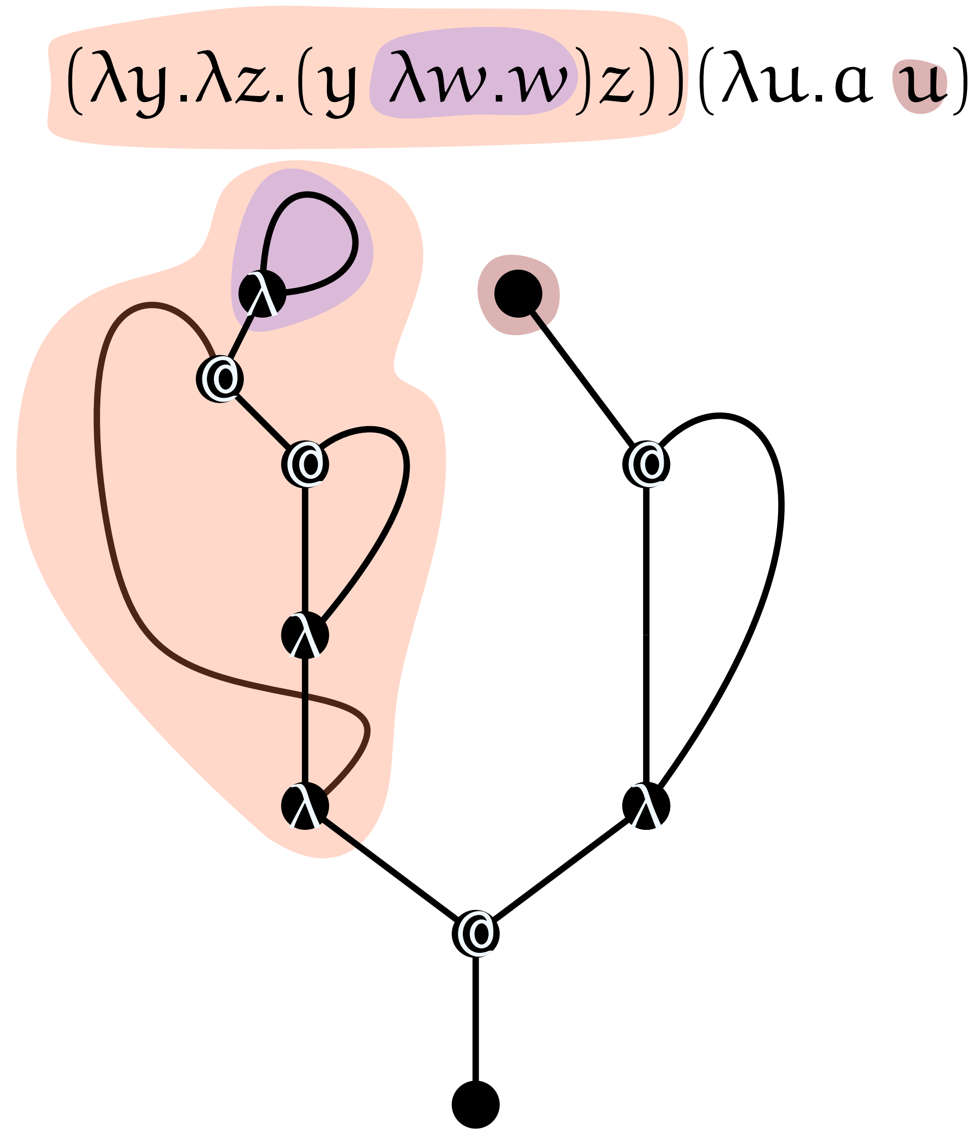

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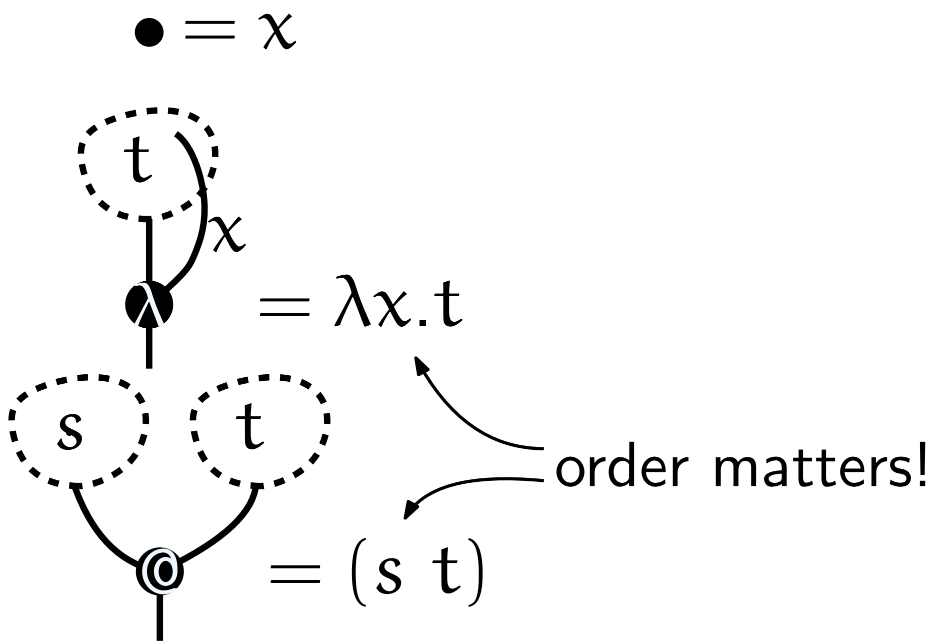

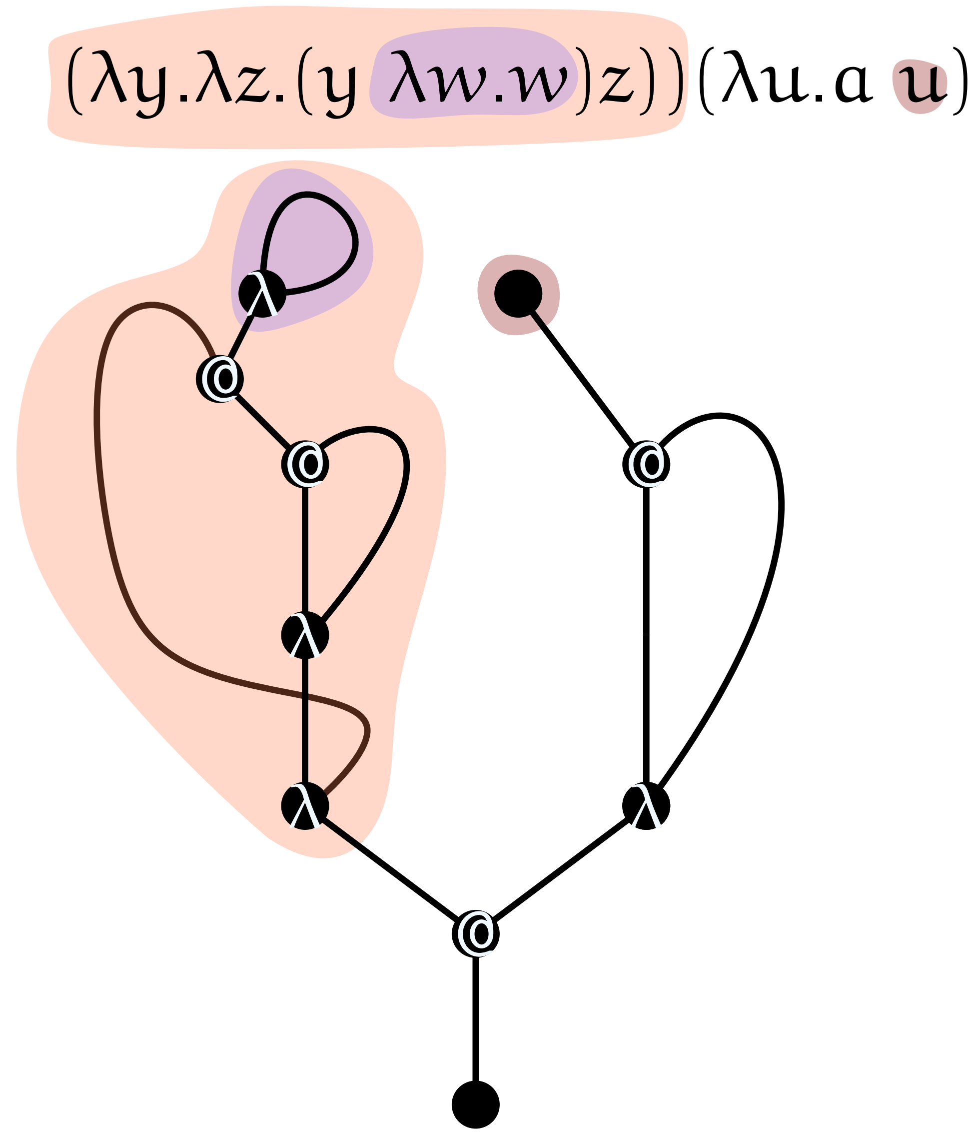

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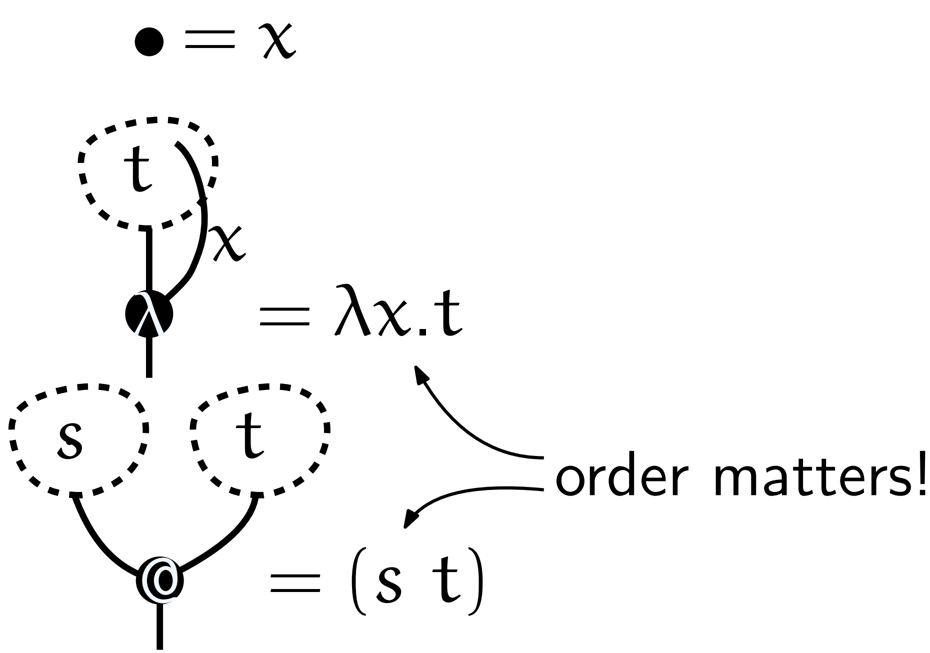

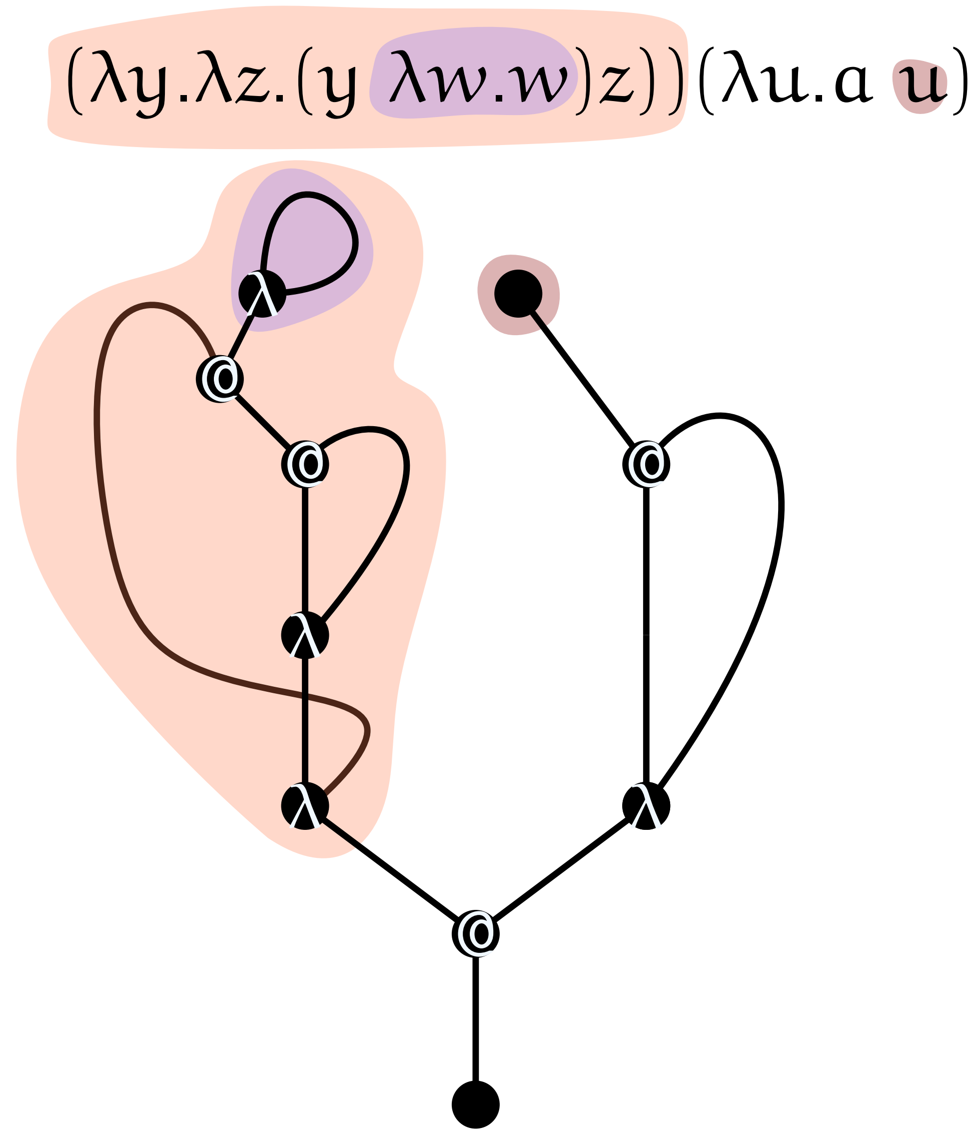

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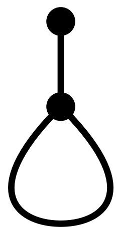
Closed linear terms ↔ trivalent maps
Closed affine terms ↔ (2,3)-valent maps
Established in [BGJ13, BGGJ13, Z16]

Why should combinatorialists be interested in λ-terms?
Decomposing rooted cubic maps



edges

$$\mathsf{T}(z) = z^2$$

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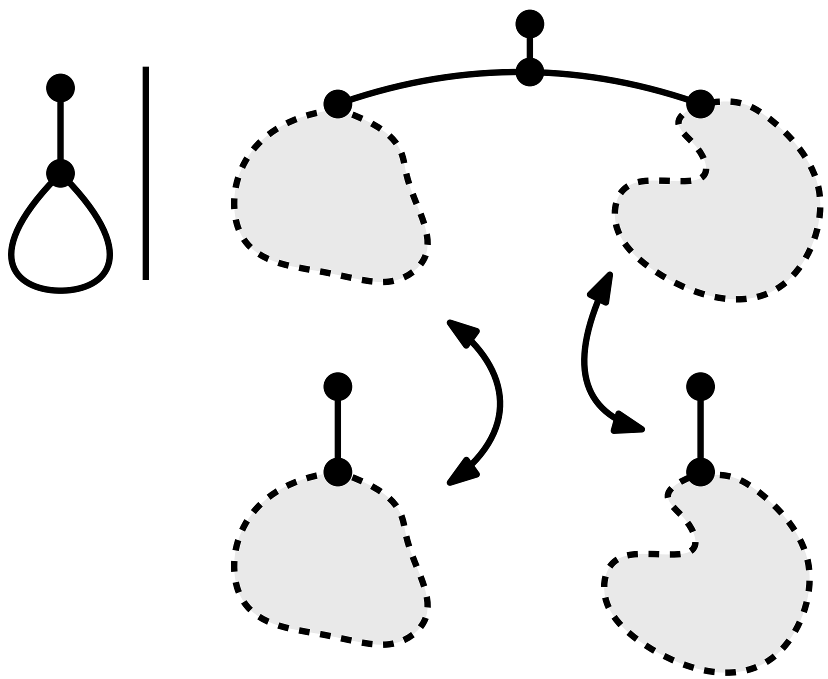

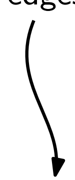

$$\mathsf{T}(z) = z^2 \ + \ z\mathsf{T}(z)^2$$

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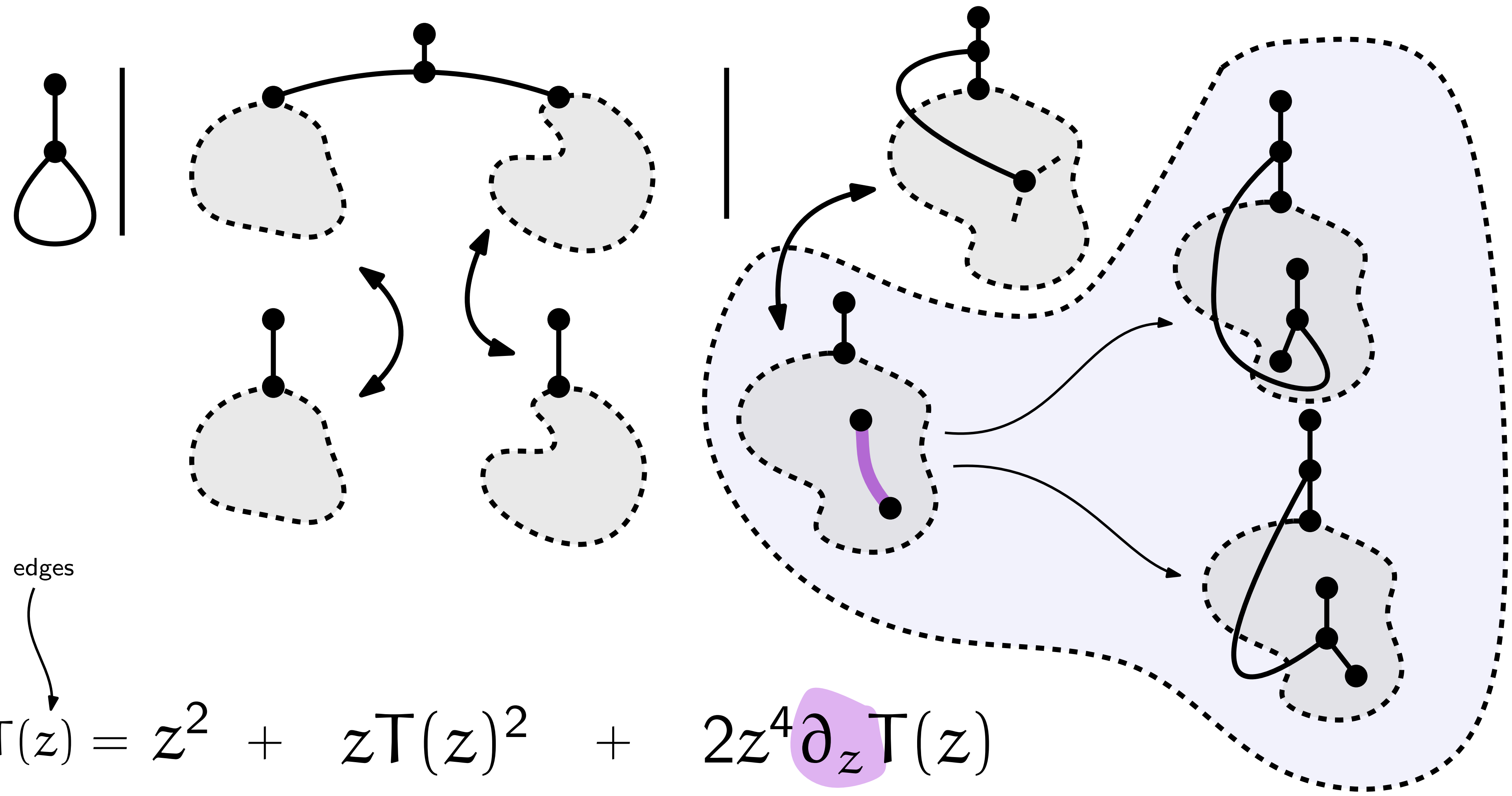
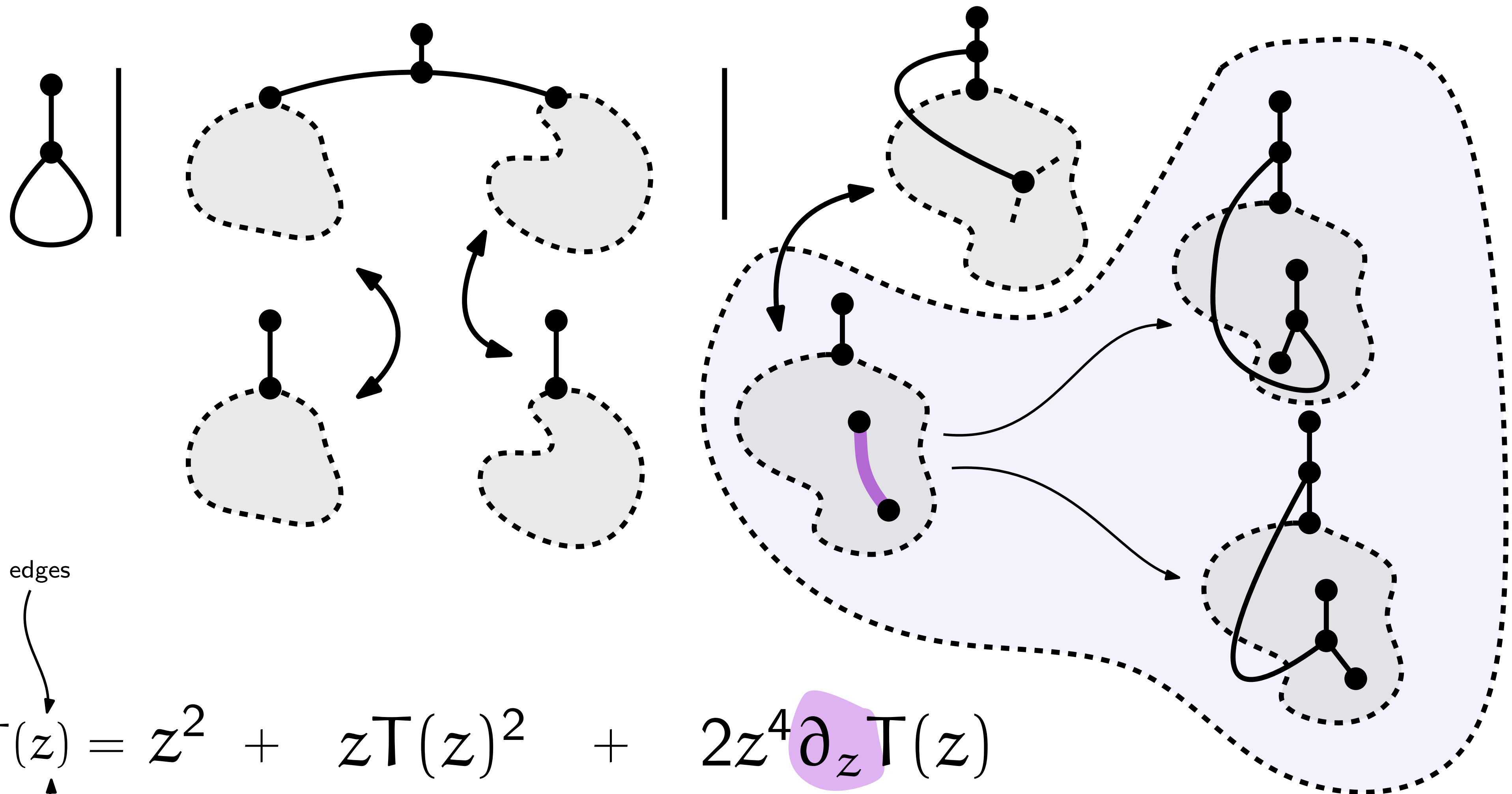


$$T(z) = z^2 + zT(z)^2 + 2z^4 \partial_z T(z)$$

Why should combinatorialists be interested in λ-terms?

Decomposing rooted cubic maps and closed linear terms!

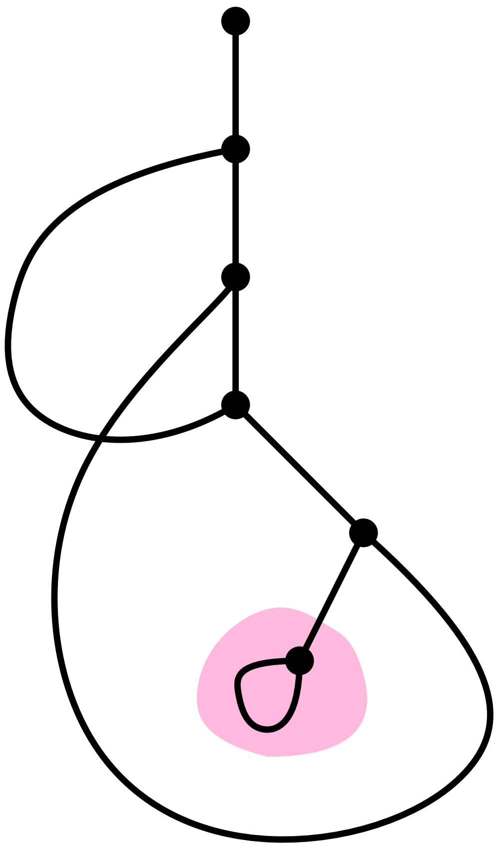


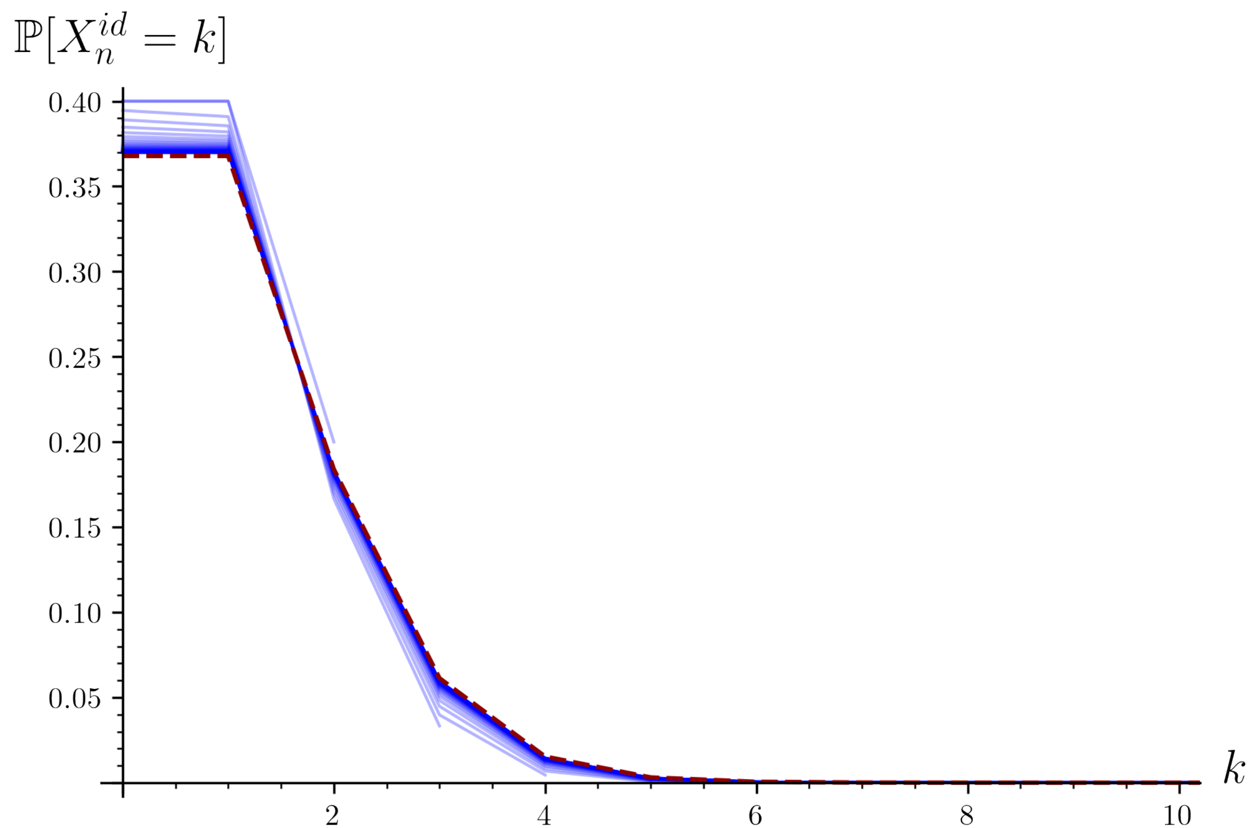
$$T(z) = z^2 + zT(z)^2 + 2z^4 \partial_z T(z)$$

$$\text{lin.term} = \lambda x.x \;\Big|\; (s\ t) \;\Big|\; \begin{array}{l}\lambda x.t[u := (x\ u)] \text{ or} \\ \lambda x.t[u := (u\ x)]\end{array}$$

Some of our previous results: limit distributions

loops = # "$\lambda x.x$"



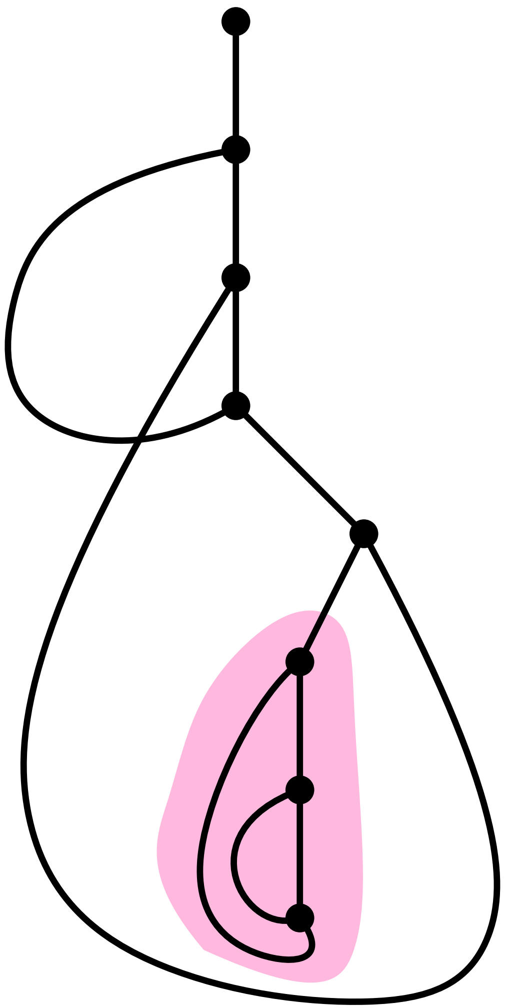


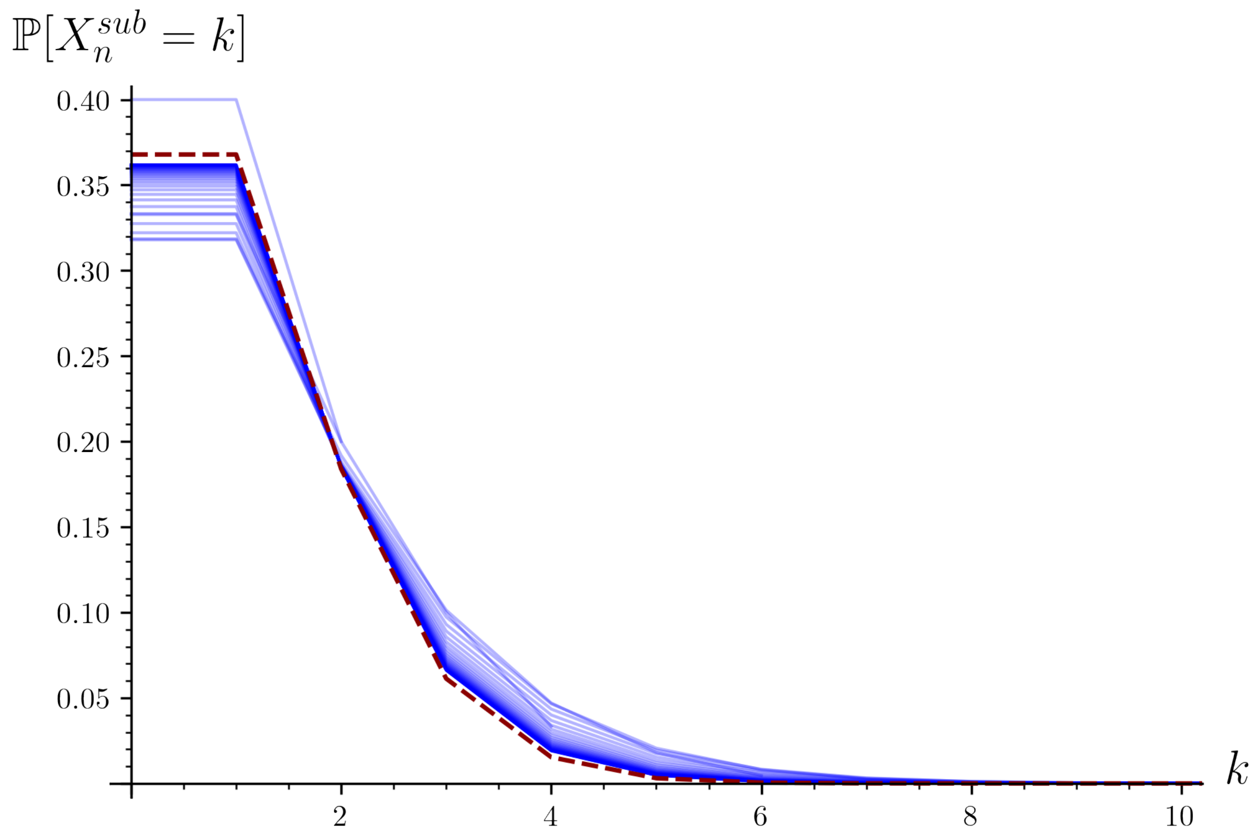
$$\lambda x.\lambda y.(y\ \lambda w.w)x$$

$$X_n^{\mathrm{id}} \xrightarrow{D} \mathrm{Poisson}(1)$$

Some of our previous results: limit distributions

bridges = # closed subterms





$\lambda x.\lambda y.(y\ \lambda z.\lambda w.zw)x$

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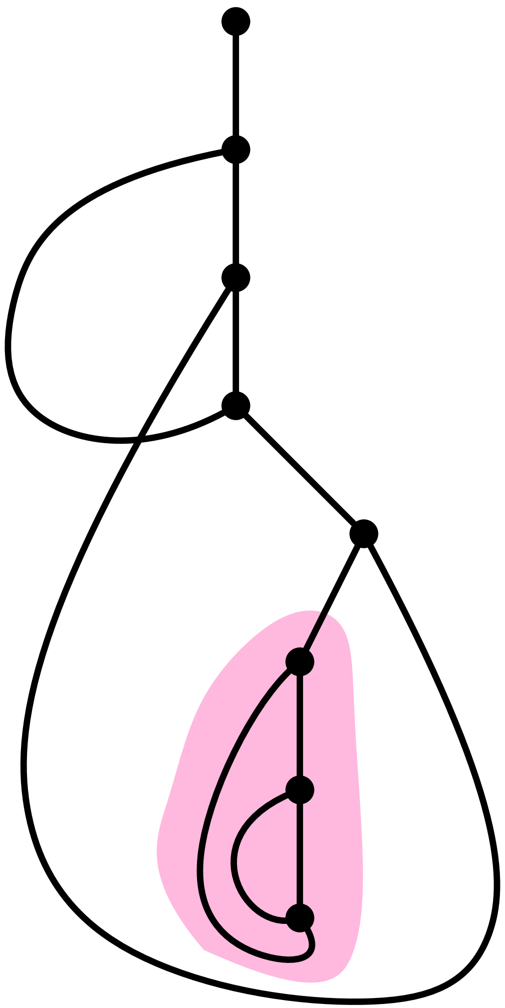
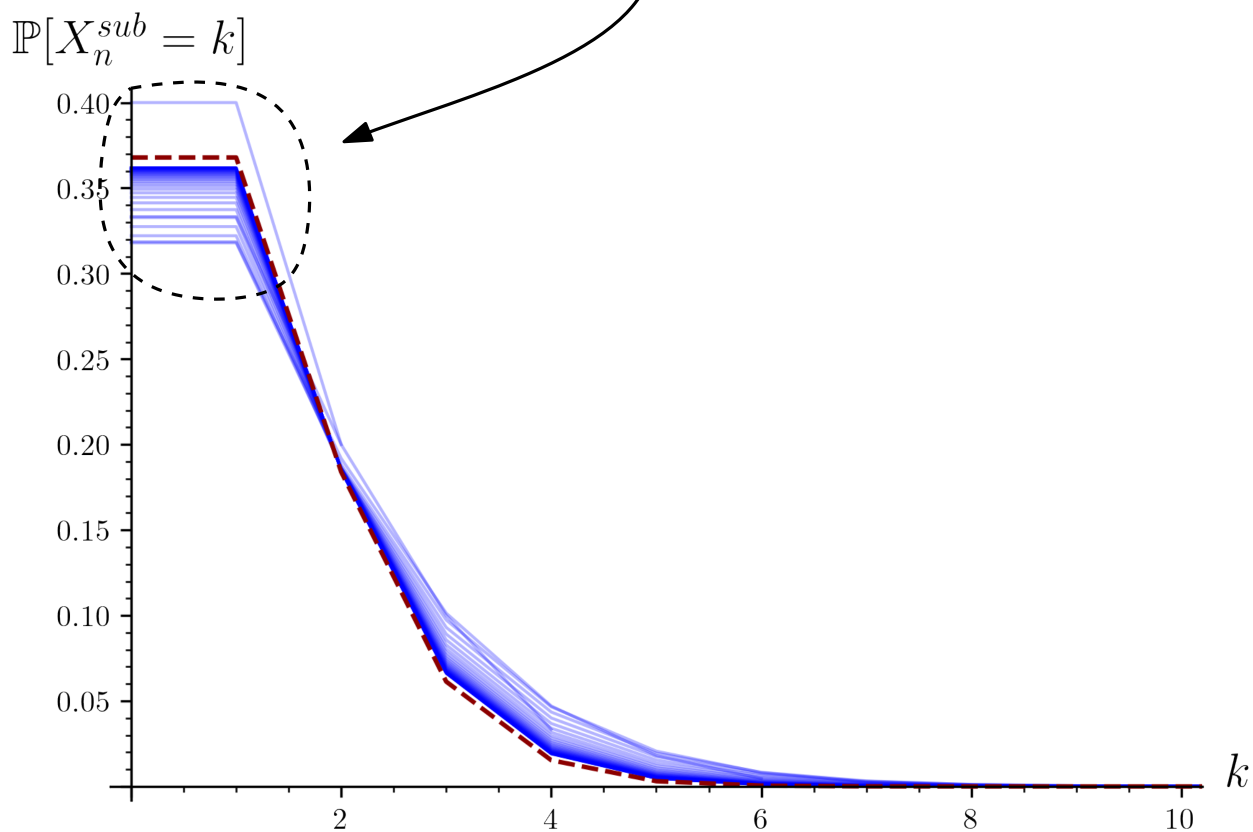


maps with one bridge ↔ bridgeless maps



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1) Track evolution of parameter in decompositions of cubic maps/λ-terms

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- Bender's theorem for compositions $F(z, G(z))$
- Coefficient asymptotics of Cauchy products

$$[z^n](A(z) \cdot B(z)) \sim a_n b_0 + a_0 b_n + O(a_{n-1} + b_{n-1})$$

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Mean number of β-redices in closed terms

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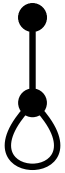
loops



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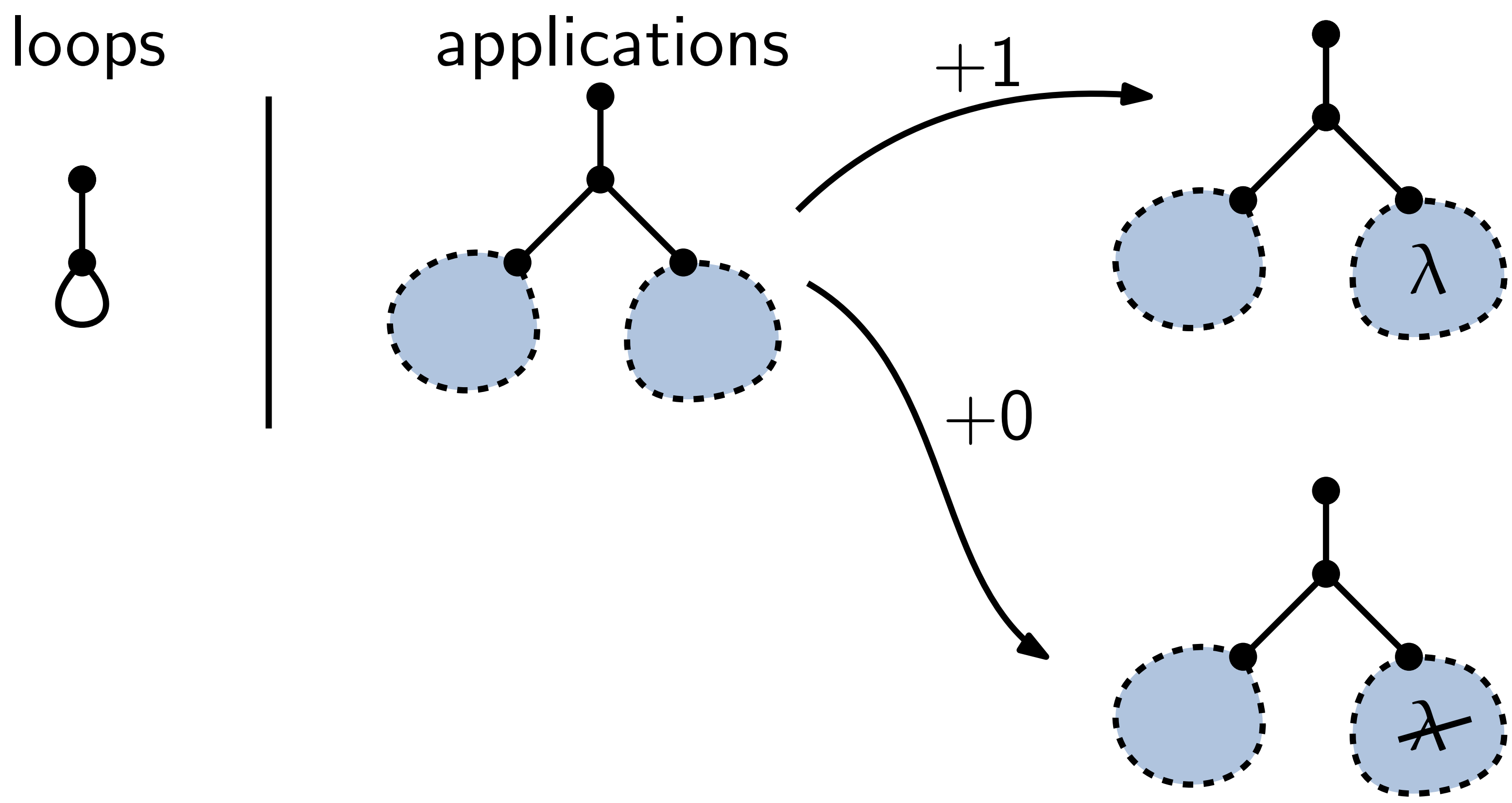

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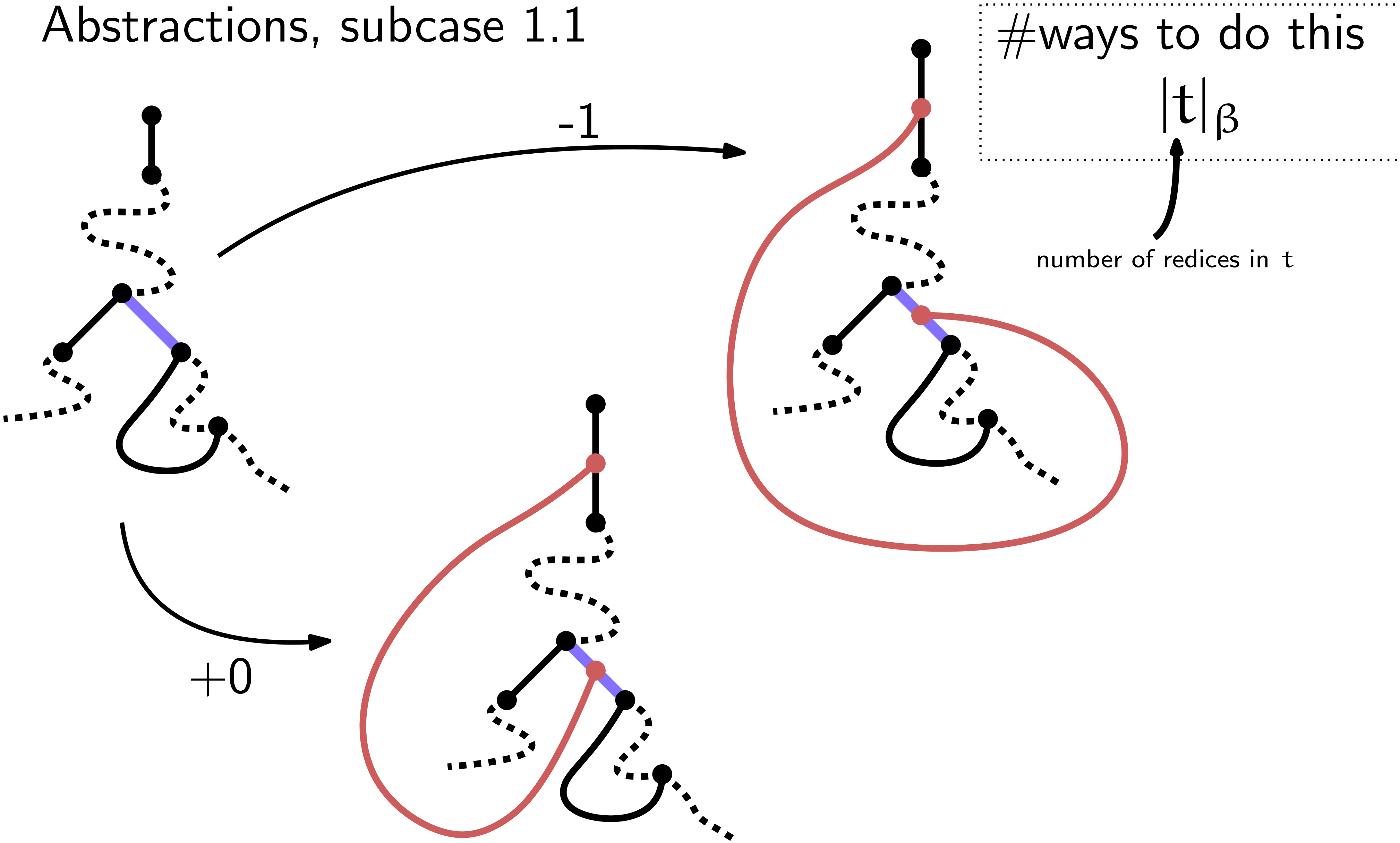
Abstractions, subcase 1.1



Mean number of β-redices in closed terms

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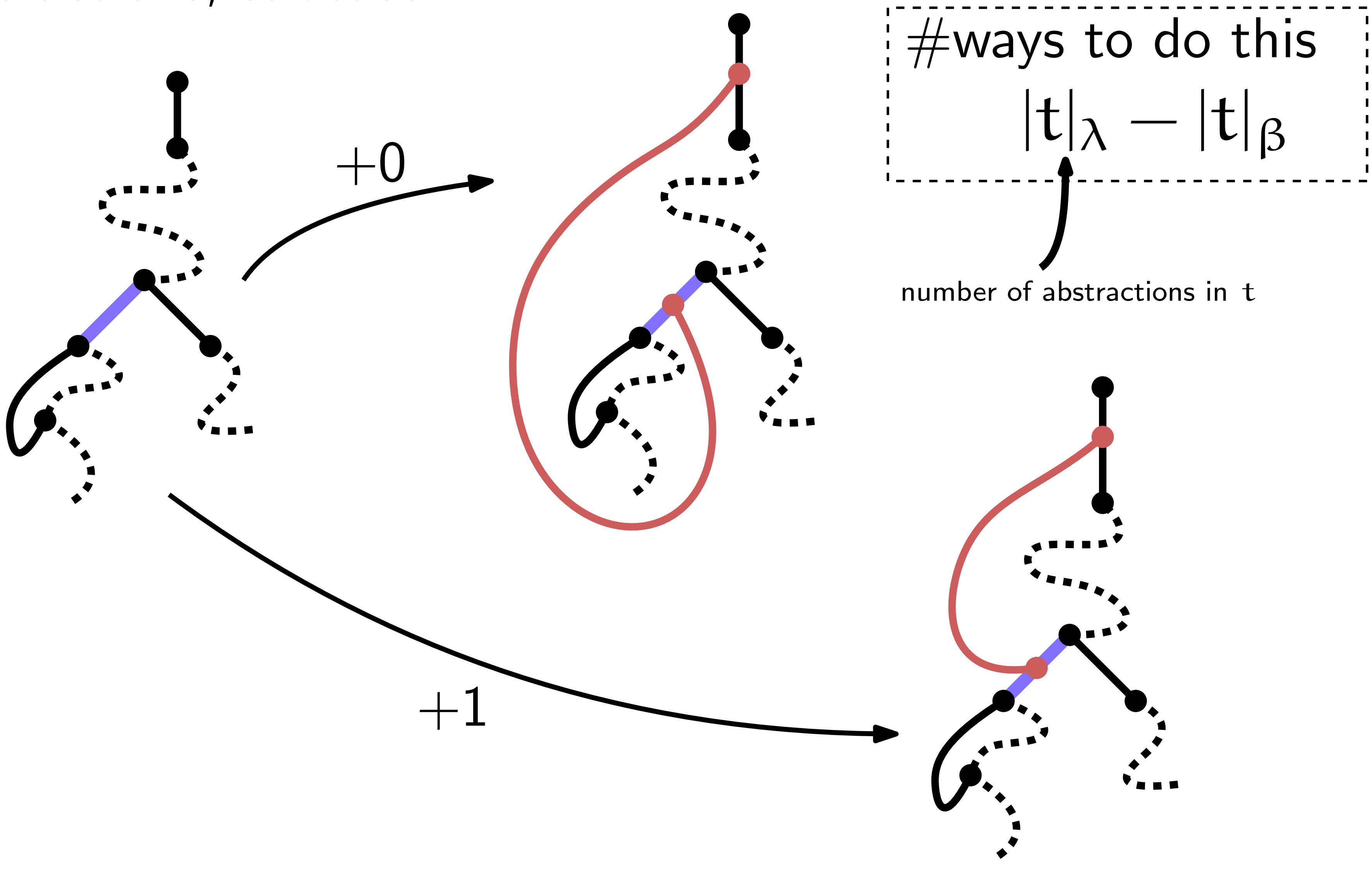
Abstractions, subcase 1.2



Mean number of β-redices in closed terms

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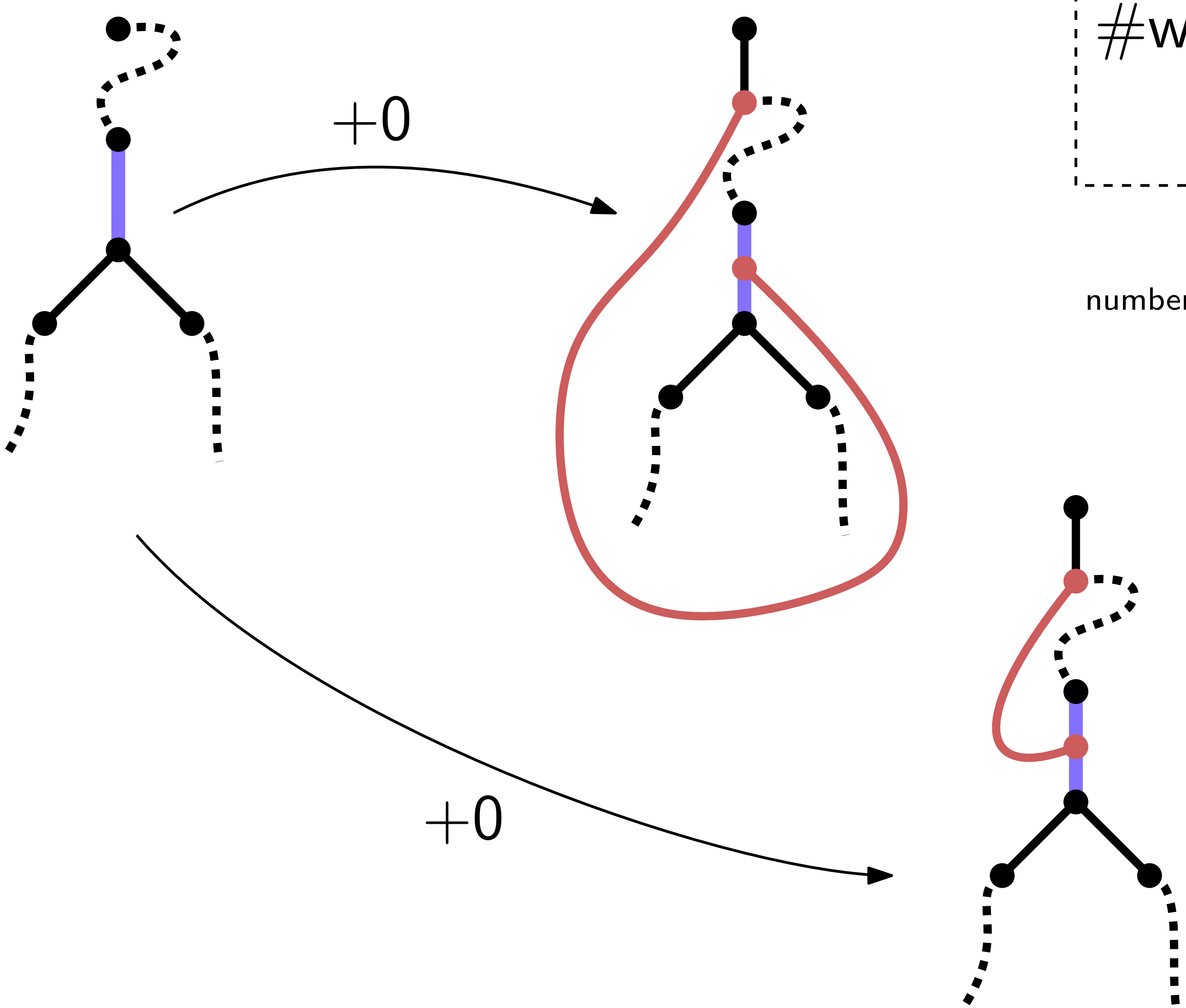
Abstractions, subcase 1.3



#ways to do this

$$|t| - |t|_\lambda$$

number of subterms in t = size of t

Mean number of β-redices in closed terms

- Building the specification of the OGF

  - $|t|_\lambda = \frac{|t|+1}{3}, |t| - |t|_\lambda = \frac{2|t|-1}{3}$
  - $r\partial_r T_0 = \sum_{t \in T_0} |t|_\beta z^{|t|} r^{|t|_\beta}$
  - $\frac{z\partial_z T_0 + T_0}{3} = \sum_{t \in T_0} \frac{|t|+1}{3} z^{|t|} v^{|t|_\beta}$
  - $\frac{2z\partial_z T_0 - T_0}{3} = \sum_{t \in T_0} \frac{2|t|-1}{3} z^{|t|} v^{|t|_\beta}$

Mean number of β-redices in closed terms

- Translating to a differential equation and pumping

$$\mathsf{T} = -z\left(z^2(\mathrm{r}+1)(1+(\mathrm{r}-1)z\mathsf{T})(\mathrm{r}-1)\partial_{\mathrm{r}}\mathsf{T} - \frac{(1+z(\mathrm{r}-1)\mathsf{T})z^3(\mathrm{r}+5)\partial_z\mathsf{T}}{3} - \frac{z^3(\mathrm{r}-1)^2\mathsf{T}^2}{3} - \frac{4z^2(\mathrm{r}-1)\mathsf{T}}{3} - z - \mathsf{T}^2\right)$$

Mean number of β-redices in closed terms

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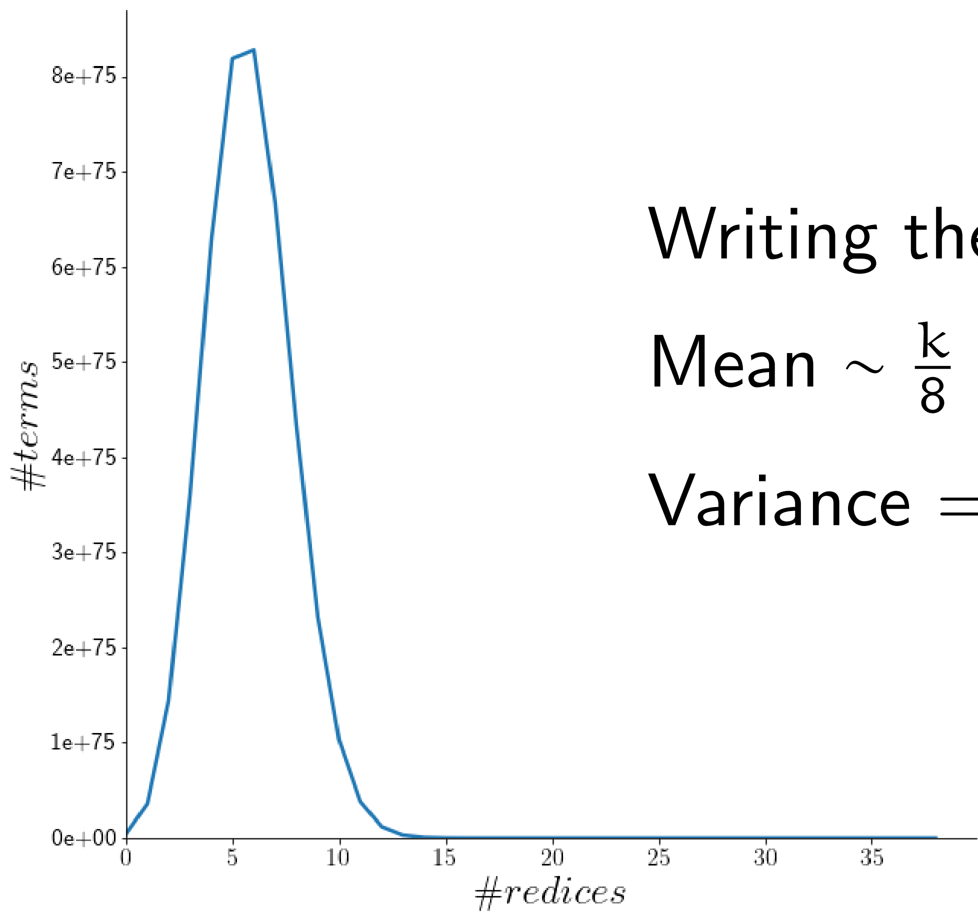
A plot of the dist. of redices for terms/maps of size $n = 119$



Writing the size as $n = 3k + 2$, we have:

Mean $\sim \frac{k}{8}$

Variance $= O(k)$

A better lower bound

A better lower bound

- Consider the following three patterns of redices

$$(\lambda x.C[(x\ u)])(\lambda y.t_2) \quad (p_1) \qquad ((\lambda x.\lambda y.t_1)t_2)t_3 \quad (p_2)$$

$$(\lambda x.x)(\lambda y.t_1)t_2 \quad (p_3)$$

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$$(\lambda x.x)(\lambda y.t_1)t_2 \quad (p_3)$$

- These are the only patterns whose reduction leaves the number of redices invariant.

- Gives a lower bound on the number of steps to reach normal form:

$$\#steps \geqslant |t|_\beta + |t|_{p1} + |t|_{p_2} + |t|_{p_3}$$

Enumerating $\mathfrak{p}_1$-patterns

- Tracking the creation/destruction of patterns during the recursive decomposition:

Enumerating p_1-patterns

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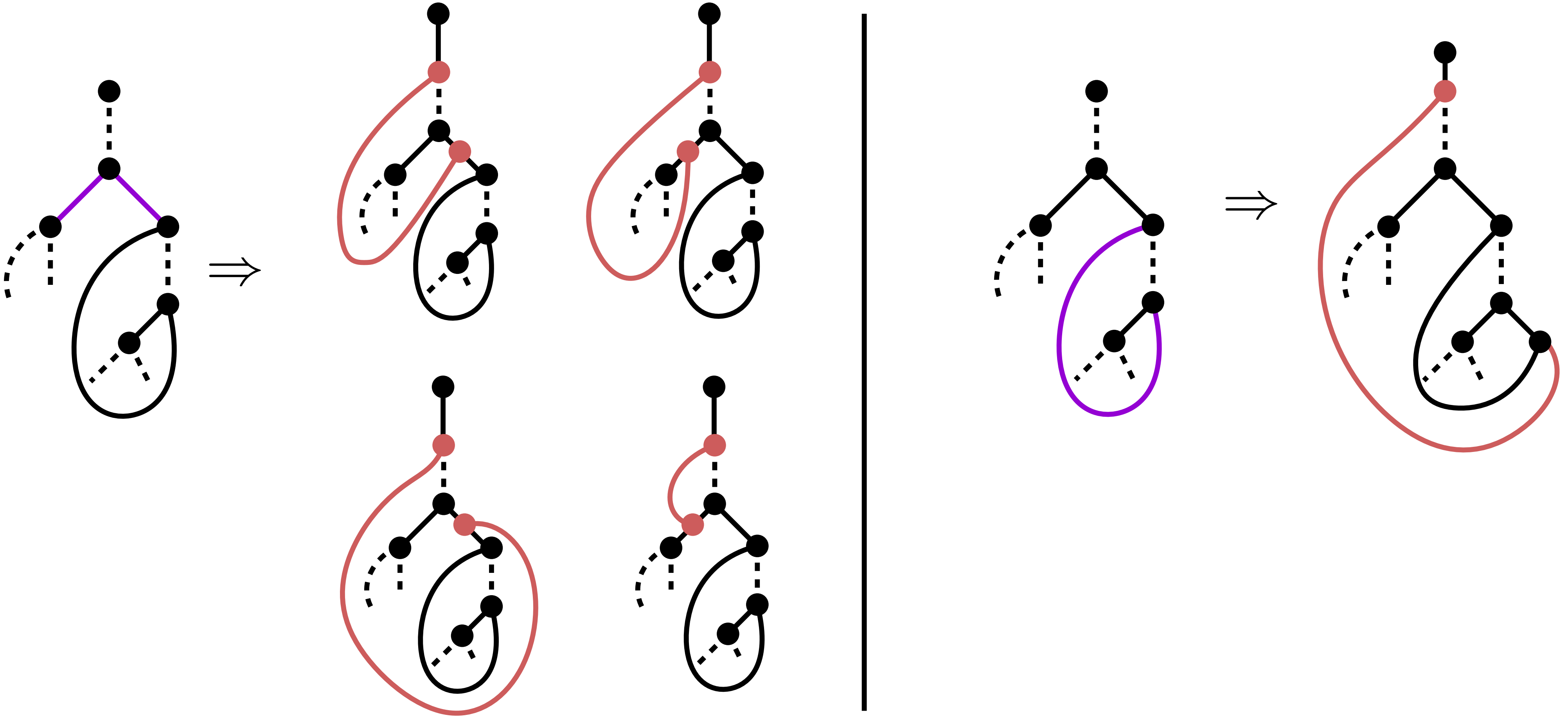
  Cuts destroying a p_1-pattern:



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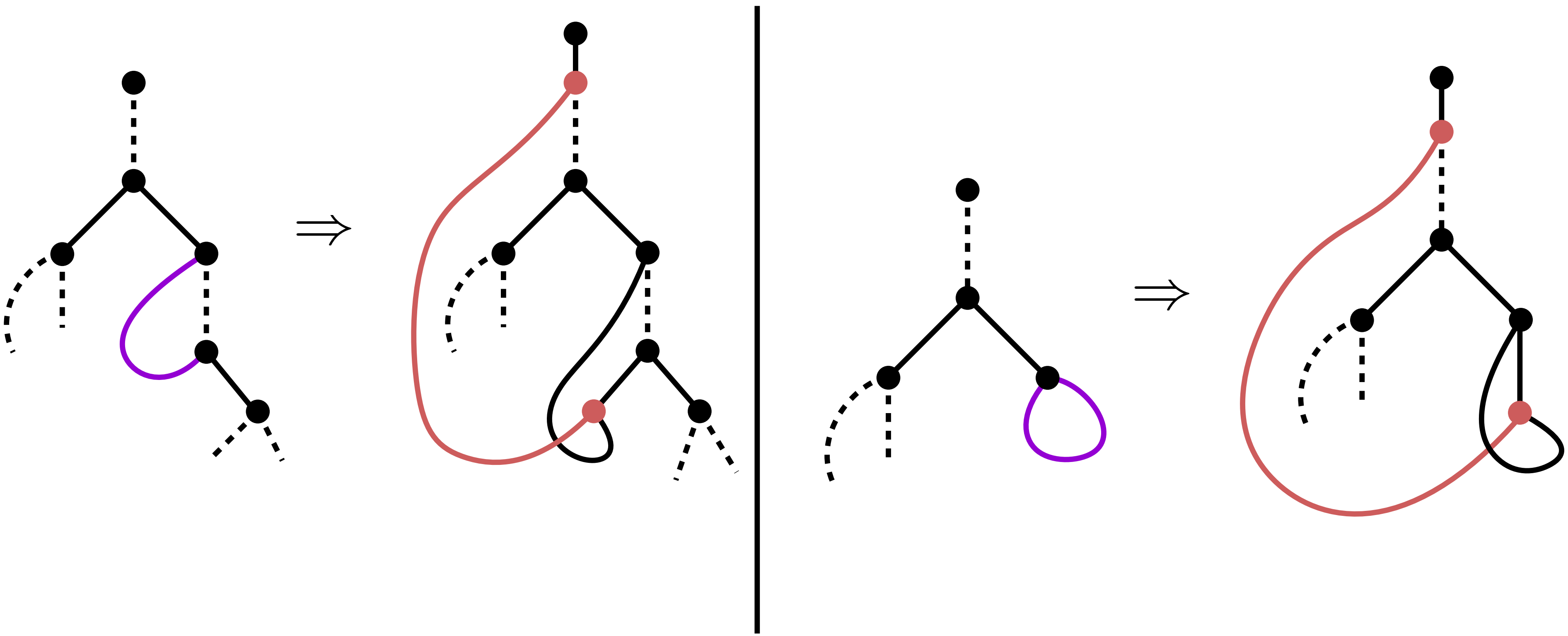
- Tracking the creation/destruction of patterns during the recursive decomposition:

  Cuts creating a p_1-pattern:



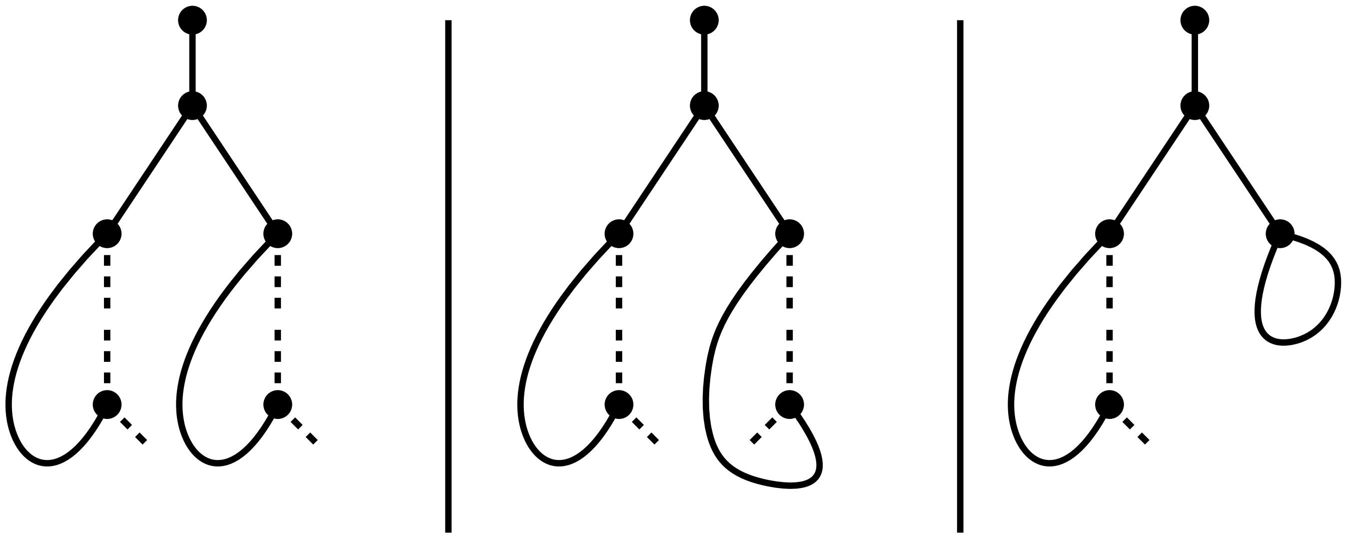
Thus we also need to keep track of:

$$C_1[\lambda x.C_2[(t_1\ x)]](\lambda y.t_2)] \quad C_1[(\lambda x.x)(\lambda y.t_2)]$$

Enumerating p_1-patterns

- Tracking the creation/destruction of patterns during the recursive decomposition:

  Applications creating p_1 and auxilliary patterns:



Thus, for an app. of the form $(l_1\ \lambda y.t_1)$ we need to consider how l_1 was formed.

Enumerating p_1-patterns

- Thus we have the following equations:

$$S = \Lambda + A$$

$$\Lambda = z^2 + 2z^4 S_z + (v - u + 4(1-u))z^3 S_u + (u - v + 4(1-v))z^3 S_v$$

$$A = zS^2 + (u-1)z(z^4 S_z + (v - u + 2(1-u))z^3 S_u + 2(1-v)z^3 S_v) \cdot \Lambda$$
$$+(v-1)z(z^2 + z^4 S_z + (u - v + 2(1-v))z^3 S_u + 2(1-u)z^3 S_u) \cdot \Lambda$$

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$$\partial_u S|_{u=1,v=1}$$
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bijection: $\partial_v \leftrightarrow \partial_u$

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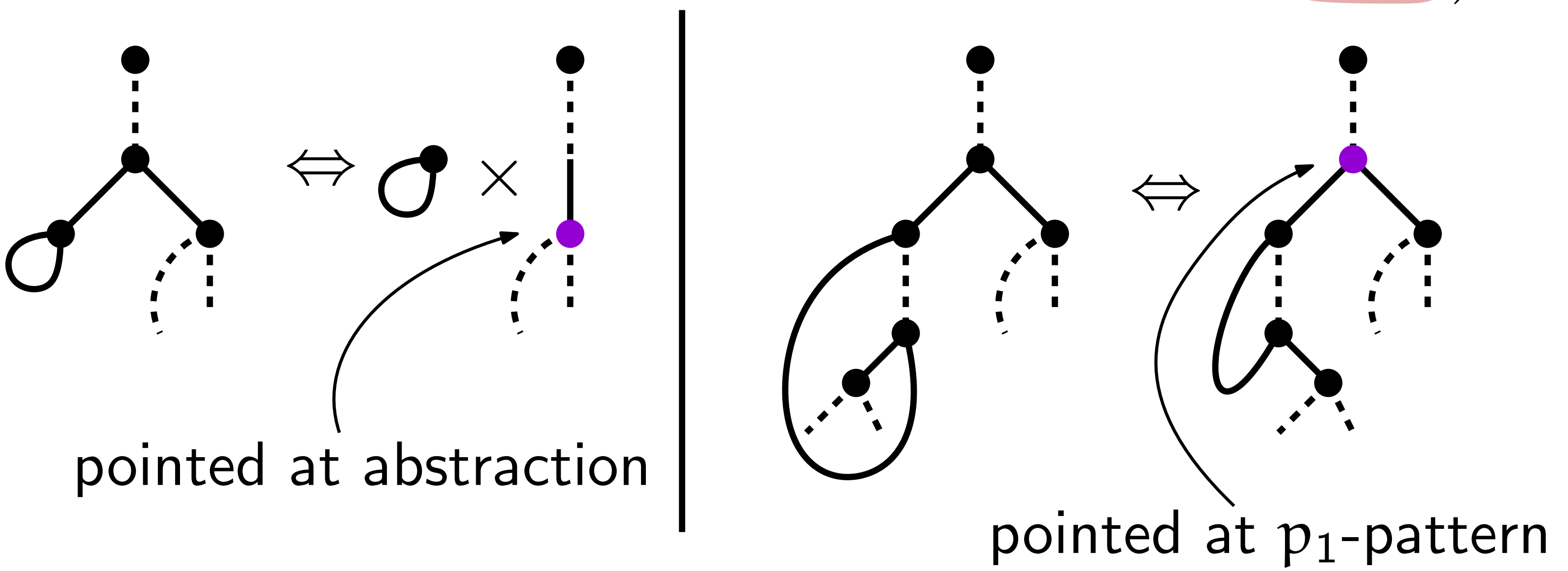

Enumerating $\mathfrak{p}_1$-patterns

- Finally we obtain a mean number of occurences:

$$\mathbb{E}[\# \ \mathfrak{p}_1 \text{ patterns}] \sim \frac{1}{6}$$

Enumerating p_1-patterns, p_2-patterns, and p_3-patterns

- Finally we obtain a mean number of occurences:

$$\mathbb{E}[\#\ p_1 \text{ patterns}] \sim \frac{1}{6}$$

- Analogously, we have a mean number of occurences for p_2:

$$\mathbb{E}[\#\ p_2 \text{ patterns}] \sim \frac{1}{48}$$

Both are asymptotically constant in expectation!

- Via different methods, we obtain:

$$\mathbb{E}[\#\ p_3 \text{ patterns}] \geqslant \frac{n}{240}$$

Asymptotically linear in n!

Conclusion

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which is quite close to Noam Zeilberger's conjecture of $\mathbb{E}[\#steps] = \frac{n}{21}$!

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Thank you!

Bonus slides!

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$$\frac{\Gamma, x, y, \Delta \vdash t}{\Gamma, y, x, \Delta \vdash t}\,\text{exc} \qquad \frac{\Gamma \vdash t}{\Gamma, x \vdash t}\,\text{wea} \qquad \frac{\Gamma, x, y \vdash t}{\Gamma, x \vdash t[y := x]}\,\text{con}$$

Computing with the λ-calculus

- Substitution rule:

$$t_1[v := t_2]$$

"replace free occurences of v in t_1 with t_2"

(renaming variables in t_1 if necessary, to avoid capturing variables of t_2)

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- Dynamics of the λ-calculus: β-reductions
(λ-terms together with β-reduction are enough to encode any computation!)

$$((\lambda x.t_1)\ t_2) \overset{\beta}{\to} t_1[x := t_2]$$

Represents:

$$f = x \mapsto t_1$$

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redex

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β-reducing general terms

- β-reduction is quite complicated:
  - Reducing a redex can create new redices!

    $$((\lambda x.(x\ z))\ (\lambda y.y)) \xrightarrow{\beta} ((\lambda y.y)\ z)$$

  - Terms may never reach a normal form, their size might even increase!

    $$((\lambda x.(x\ x))(\lambda x.(x\ x\ x))) \xrightarrow{\beta} (\lambda x.(x\ x\ x))(\lambda x.(x\ x\ x))(\lambda x.(x\ x\ x))$$

  - Order in which redices are reduced matters!

$$(\lambda x.z)((\lambda x.(x\ x))(\lambda x.(x\ x))) \rightarrow (\lambda x.z)((x\ x)[x := (\lambda x.(x\ x))]) = \ldots$$

$$(\lambda x.z)((\lambda x.(x\ x))(\lambda x.(x\ x))) \rightarrow z[x := (\lambda x.x\ x)(\lambda x.x\ x)] = z$$

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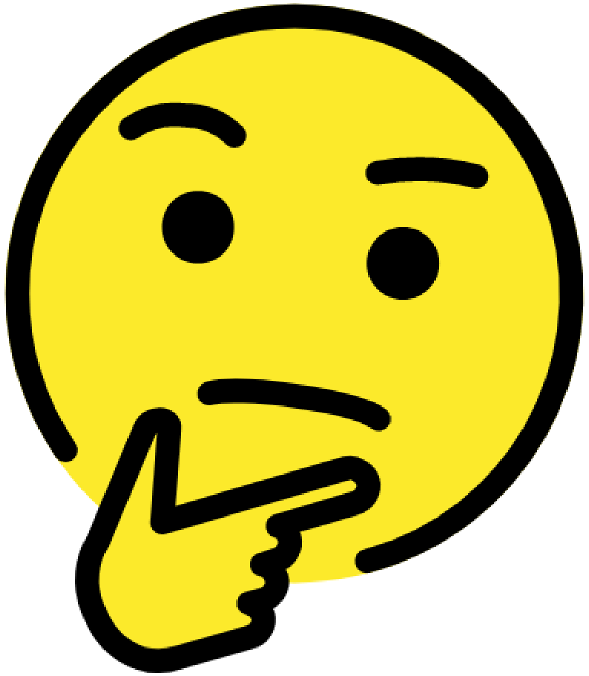

Previous work on the reduction of λ-terms

- Asymptotically almost all λ-terms are strongly normalizing. [DGKRTZ13]
  Model based on previously-presented syntax and size defined recursively as:

$$|x| = 0, |(a\ b)| = 1 + |a| + |b|, |\lambda x.t| = 1 + |t|$$

- Asymptotically almost no λ-term is strongly normalizing. [DGKRTZ13,BGLZ16]
  Model based on de Bruijn indices or combinators (together with appropriate size functions)

Parameter sensitive to the syntax
and the size of terms!

- Almost every simply-typed λ-term has a long β-reduction sequence [SAKT17]

Subfamilies of λ-terms

General terms: no restrictions on variable use

$\lambda x.\lambda y.x$

$\lambda x.\lambda y.x\ (y\ a)$

$(\lambda x.xx)(\lambda y.yy)$

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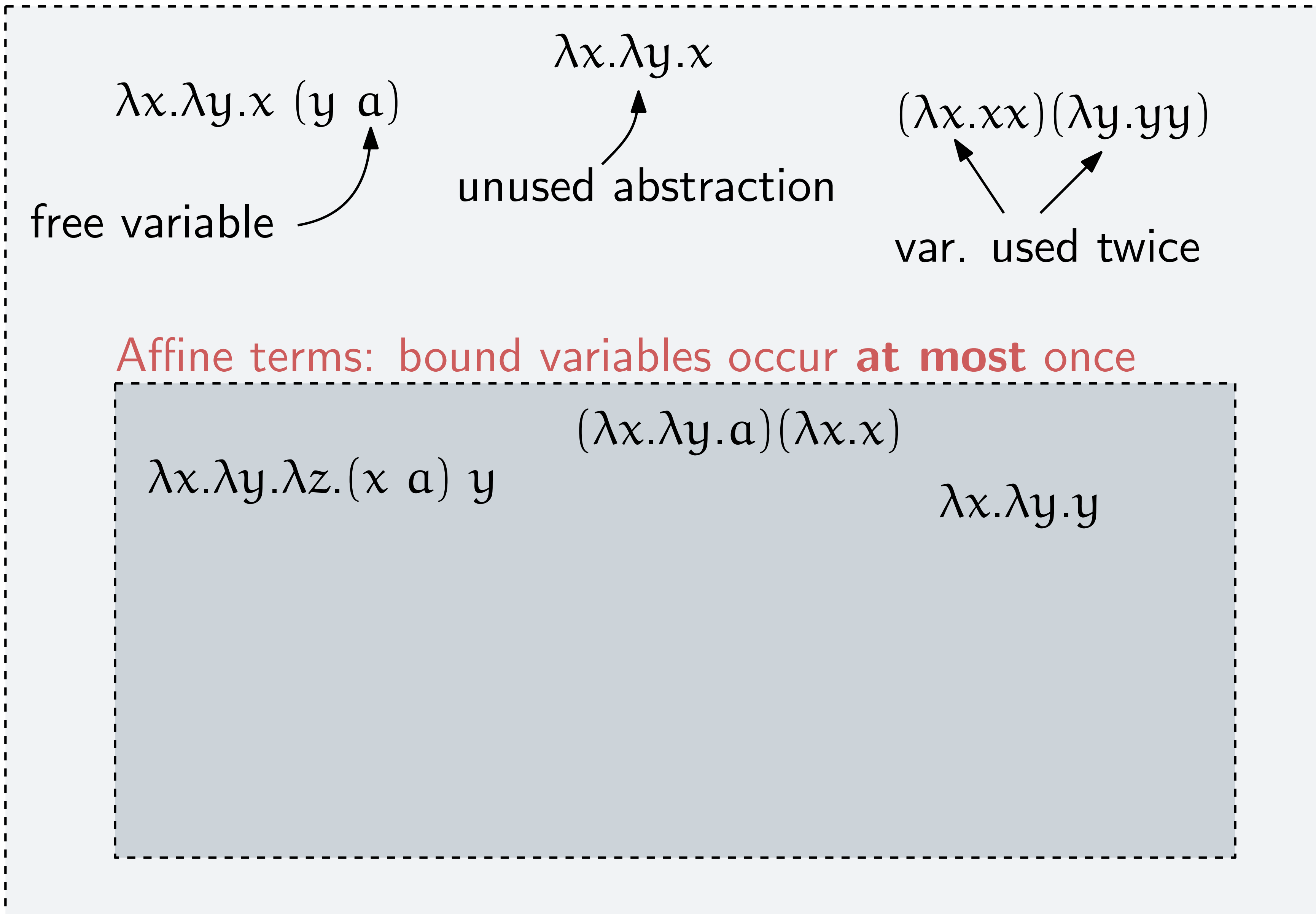


affine = no contraction

Subfamilies of λ-terms

General terms: no restrictions on variable use

$\lambda x.\lambda y.x\ (y\ a)$ — free variable

$\lambda x.\lambda y.x$ — unused abstraction

$(\lambda x.xx)(\lambda y.yy)$ — var. used twice

Affine terms: bound variables occur **at most** once

$\lambda x.\lambda y.\lambda z.(x\ a)\ y$

$(\lambda x.\lambda y.a)(\lambda x.x)$

$\lambda x.\lambda y.y$

Linear terms: bound variables occur **exactly** once

$\lambda x.\lambda y.(y\ x)a$

$\lambda x.\lambda y.(y\ a)(b\ x)$

$\lambda x.a(\lambda z.(\lambda y.y\ (x\ z)))$

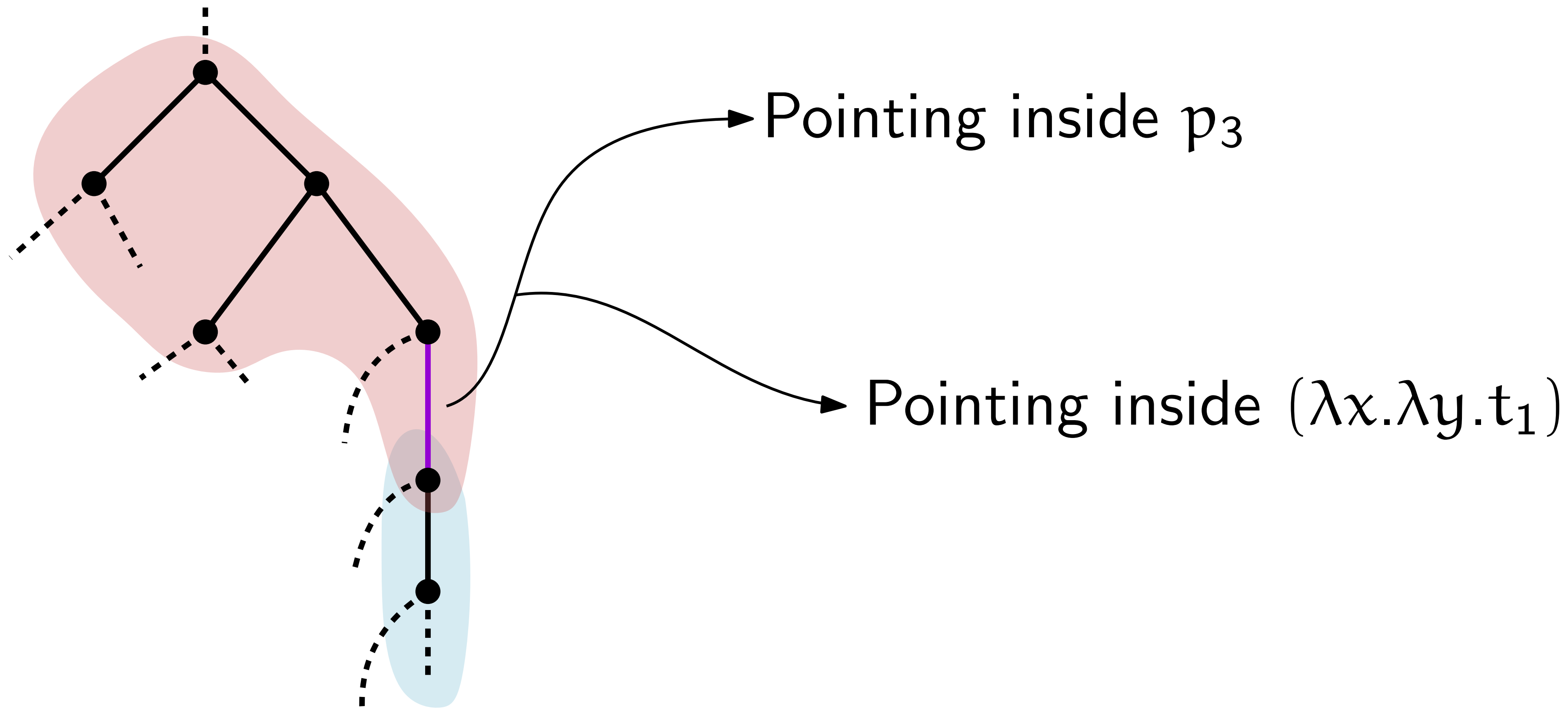
affine = no contraction linear = no contraction, no weakening

Enumerating p_3-patterns

- As before, we'll also need to enumerate auxilliary patterns:

$(\lambda x.\lambda y.t_1)$ $\quad$ $(\lambda x.\lambda y.t_1)\ t_2$ $\quad$ $(\lambda x.\lambda y.t_1)\ t_2\ t_3 \quad (p_3)$

- However we run into a problem:



Enumerating p_3-patterns

- Generatingfunctionology fails, we revert to more elementary methods:

$$\mathbb{E}(V_n) = \mathbb{E}(V_n|\Lambda_n) \cdot \frac{|\Lambda_n|}{|L_n|} + \mathbb{E}(V_n|A_n) \cdot \frac{|A_n|}{|L_n|}$$

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Magic: linear over *families* of all possible abstractions created via cuts from a fixed term!

$$\overline{X}_n = (2n - 12)\overline{X}_{n-3} 2\overline{Y}_{n-3}$$
$$\overline{Y}_n = (2n - 6)Y_{n-3} - 6Y_{n-3}$$
$$\overline{Z}_n = 2(n - 4)(Z + \mathbf{1}_{\Lambda_n})$$

where: X_n counts # of p_1 patt. over terms of size n

Y_n is the same for the pattern $(\lambda x.\lambda y.t_1)\ t_2$, and

Z is the same for the pattern $(\lambda x.\lambda y.t_1)$

The $\overline{V}$ for $V \in \{X_n, Y_n, Z_n\}$ are cummulatives over families of abstractions

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