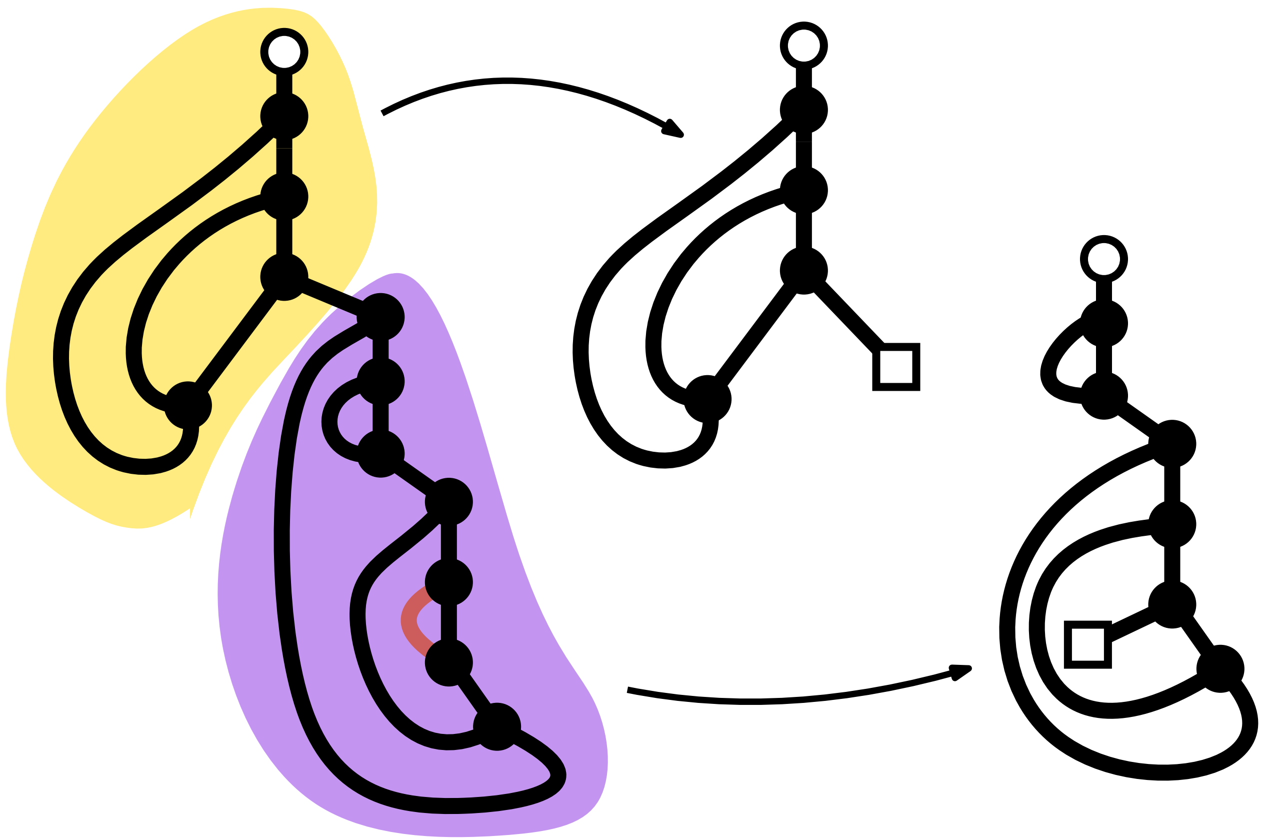
A novel interpretation of the planar Goulden-Jackson recurrence using the planar λ-calculus



Alexandros Singh (LIPN, Paris 13)

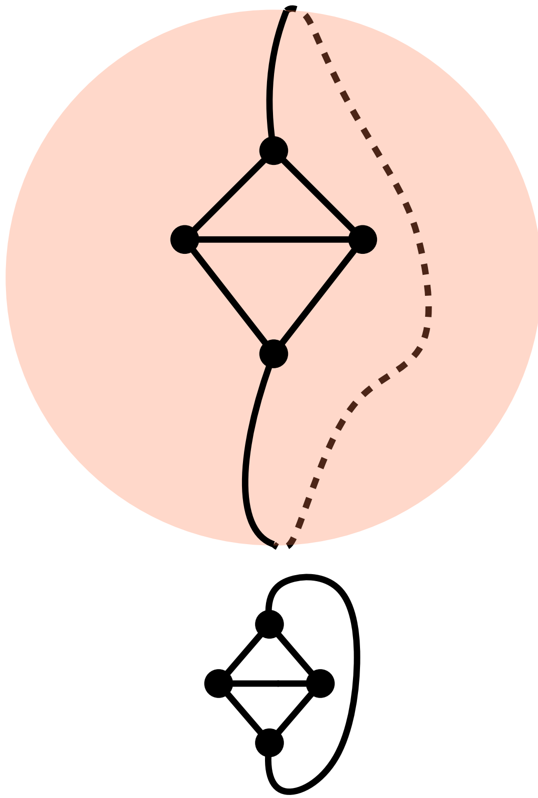
Thursday, March 16th 2023

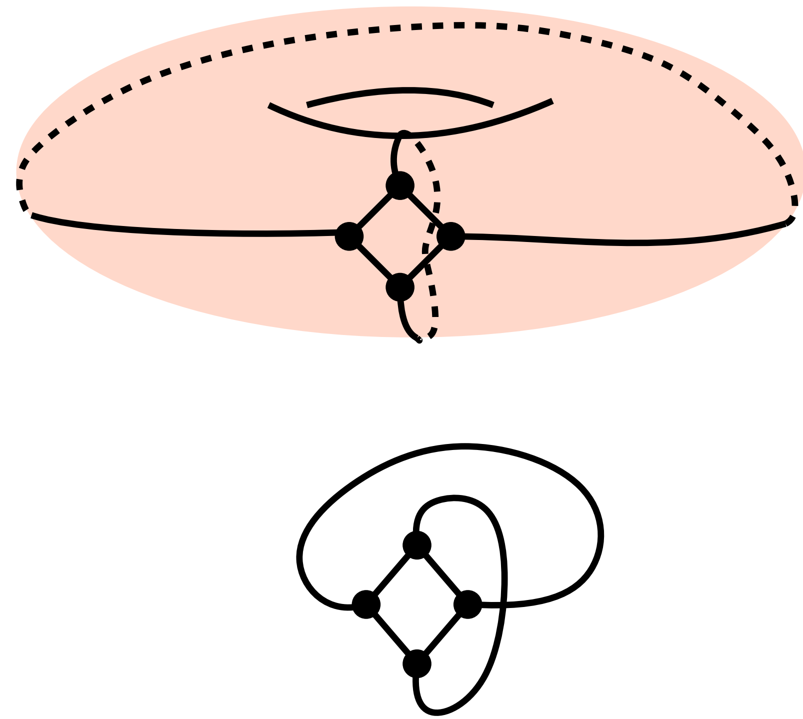
Journées ALEA 2023

The plan

- A brief overview of maps and the λ-calculus
- Context and related results
- The planar λ-calculus
- Goulden-Jackson recurrence for planar maps
- Closing remarks

What are maps?





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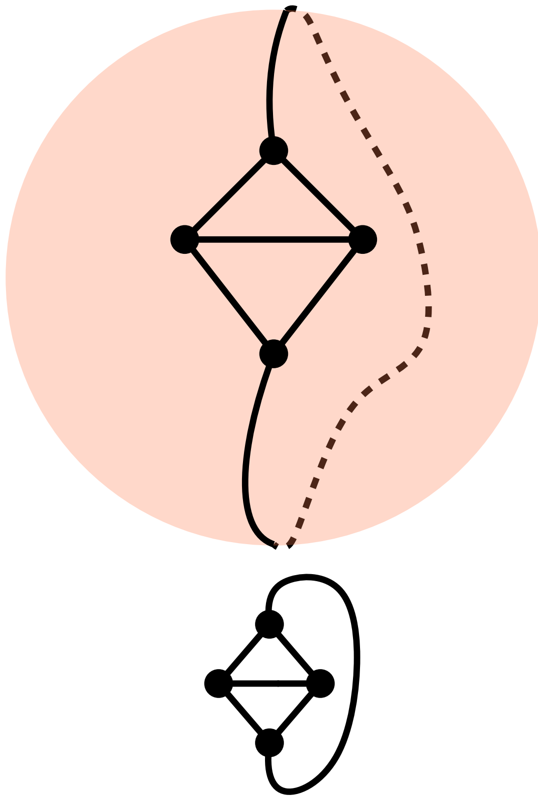

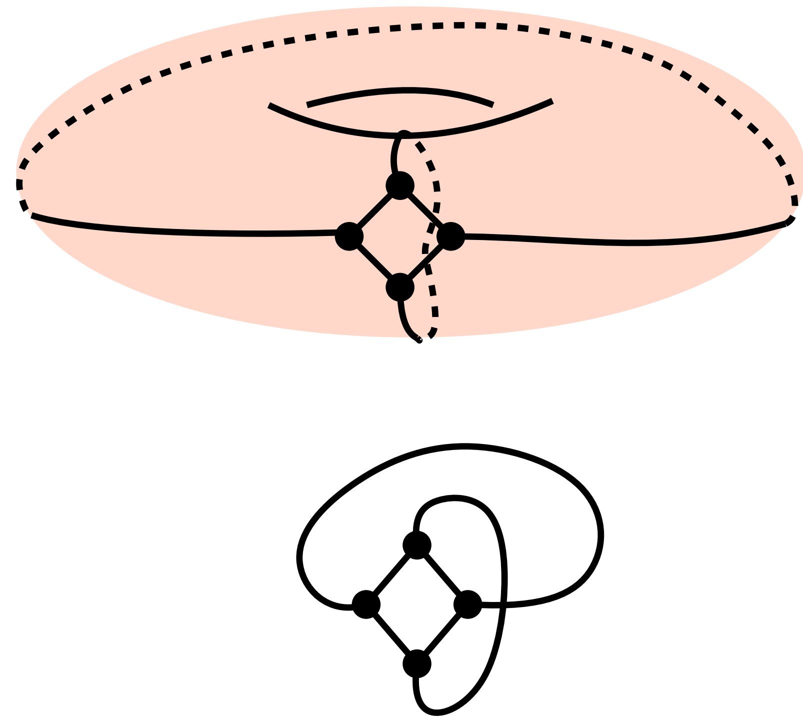

4CT...

- A central object in modern combinatorics, but not only that: probability, algebraic geometry, theoretical physics...

scaling limits... matrix integrals, Witten's conjecture, . . .

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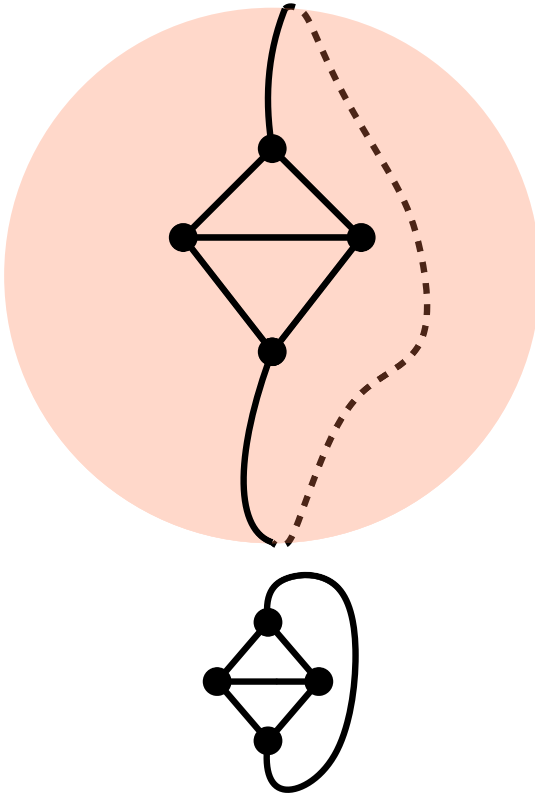

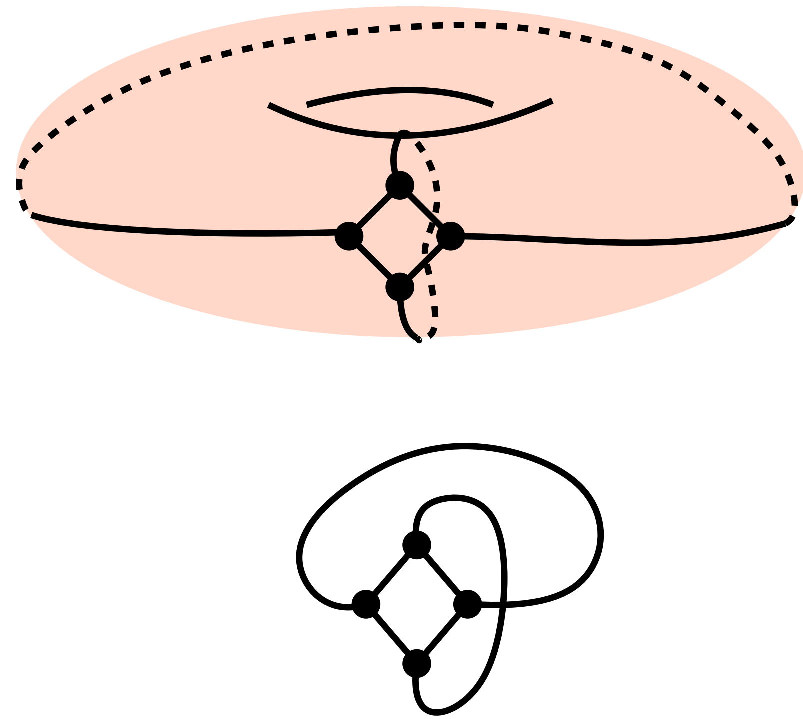

- A central object in modern combinatorics, but not only that: probability, algebraic geometry, theoretical physics...

- Their enumeration was pioneered by Tutte in the 60s, as part of his approach to the four colour theorem.

What is the λ-calculus?

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- In its typed form: functional programming, proof theory,...

Examples of λ-terms

$(\lambda x.(x\ y))$	open term
$(\lambda x.(x\ x))(\lambda z.z)$	closed term
$(y\ (\lambda x.x))$	open term with closed subterm

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Terms are considered up to *careful* renaming of variables:

$$(\lambda x.\lambda y.(x\ y\ x)) \overset{\alpha}{=} (\lambda z.\lambda y.(z\ y\ z)) \overset{\alpha}{\neq} (\lambda x.\lambda y.(z\ y\ x))$$

Linking terms and maps

- In 2013, Bodini, Gardy, Jacquot, describe a series of bijections:

rooted trivalent maps $\leftrightarrow$ closed linear terms
rooted (2,3)-valent maps $\leftrightarrow$ closed affine terms

In the same year, together with Gittenberger, they study:

$\mathrm{BCI}(\mathrm{p})$ terms (each bound variable appears p times)

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- In 2015, Zeilberger advocates for

  "linear lambda terms as invariants of rooted trivalent maps"

Some results ●=w. Bodini, Zeilberger

Parameters on maps and terms of arbitrary genus (number of):

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Similar results for planar maps/terms, plus: a new interpretation of a recurrence of Goulden and Jackson.

This talk!

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Inductive definition (keeping track of free variables):

$$\frac{}{x \vdash x}\ \text{var}$$

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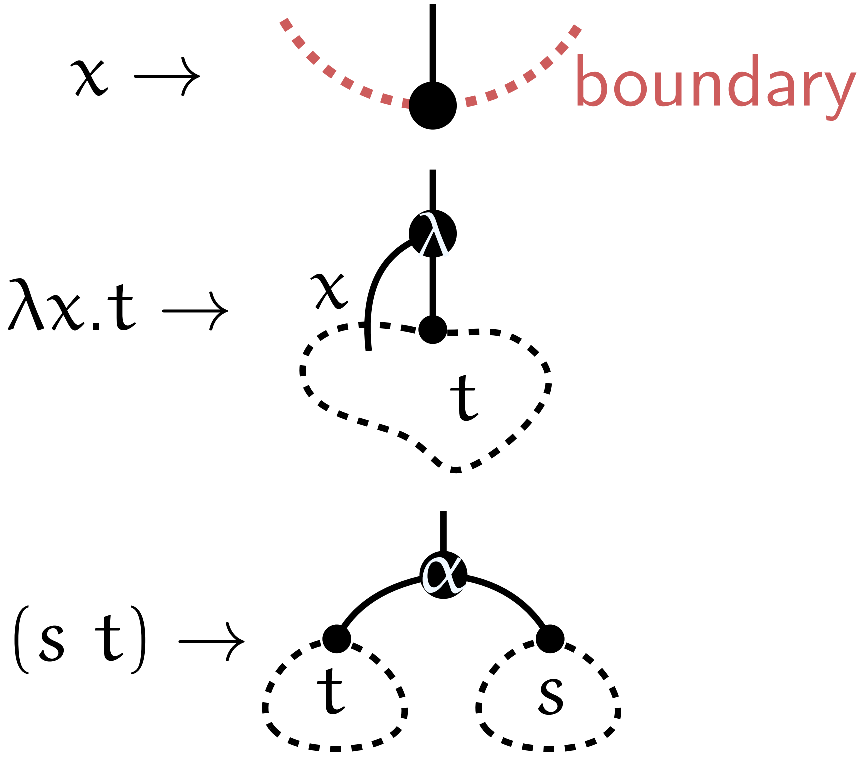
From open planar terms to maps



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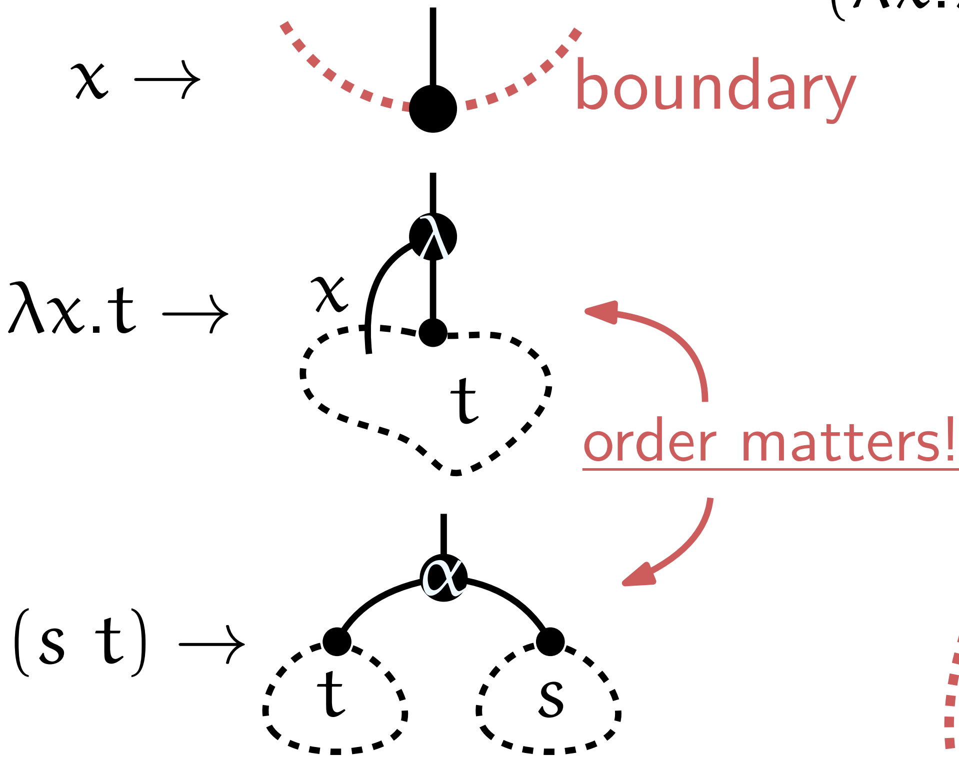

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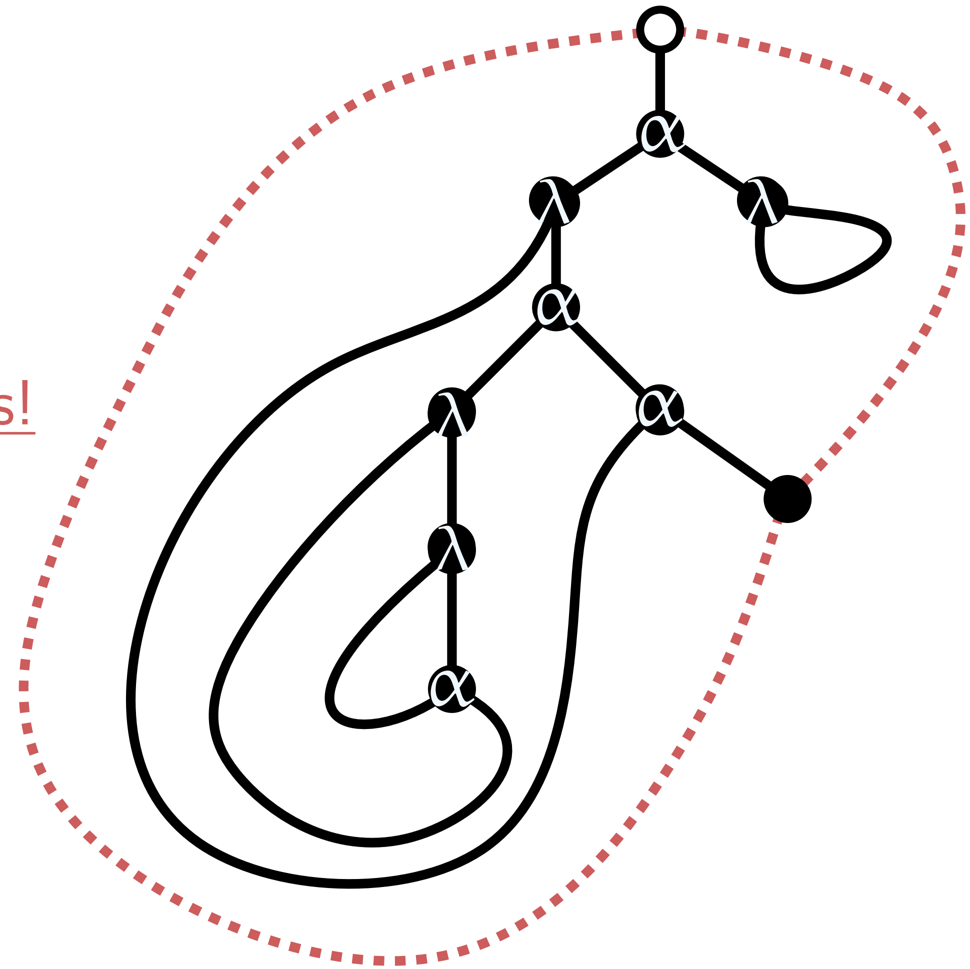

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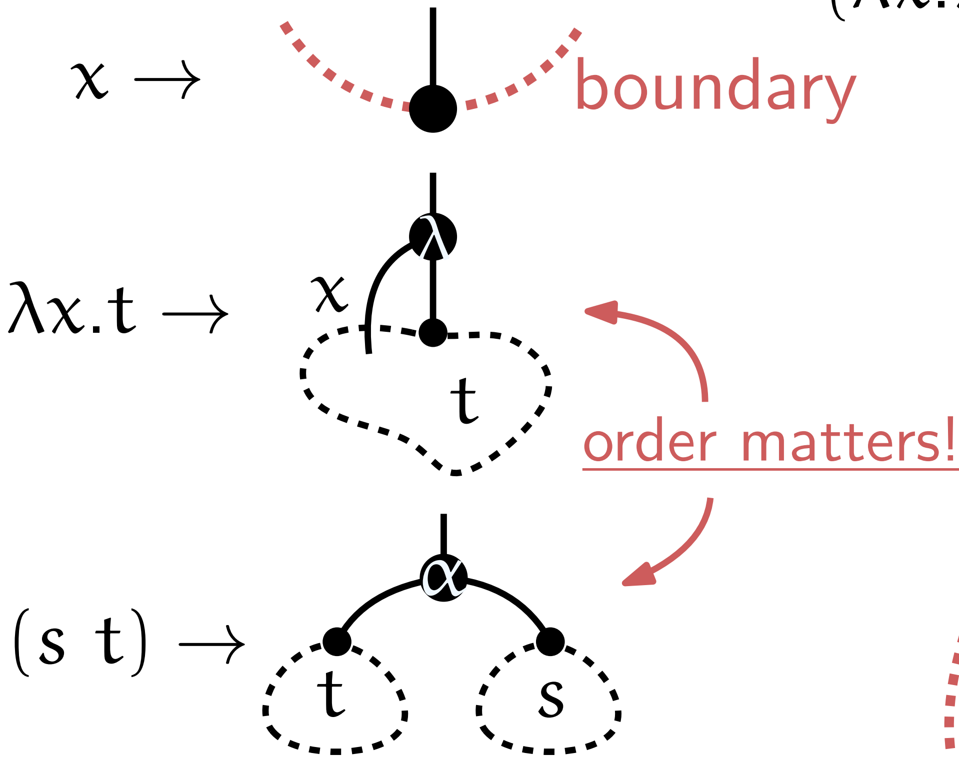

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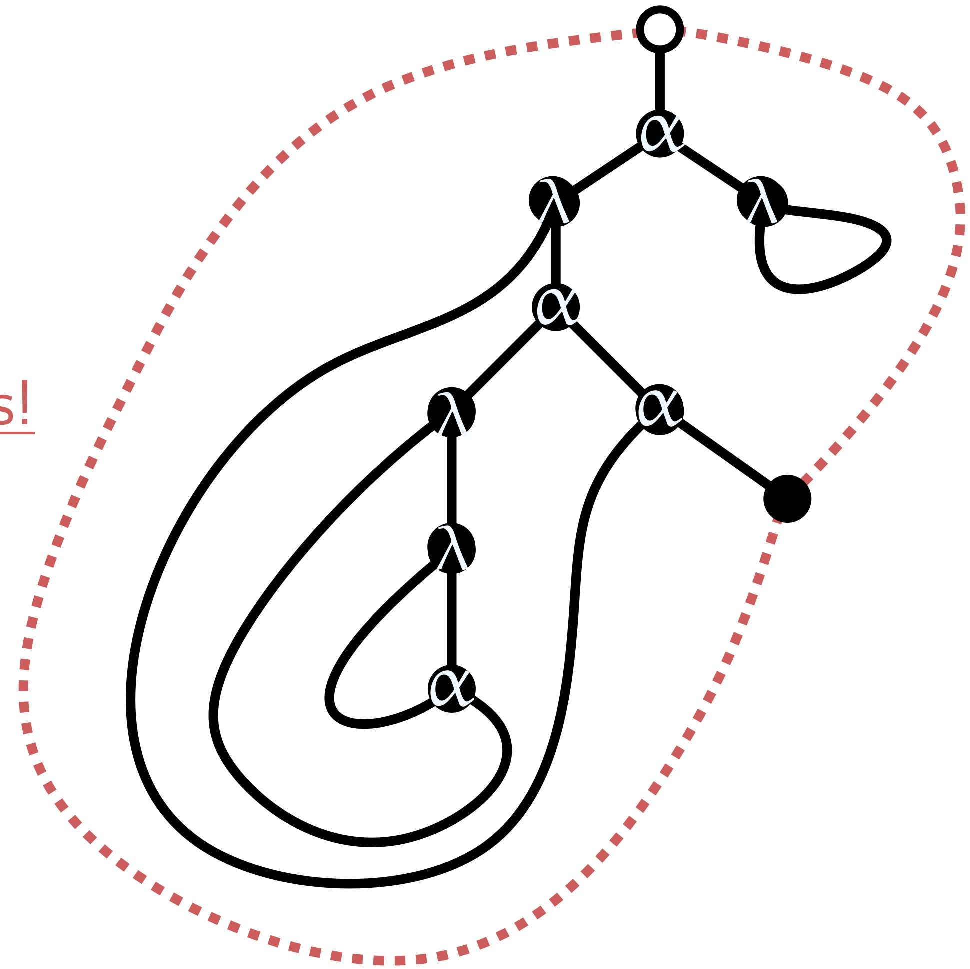

Dictionary

- # subterms ↔ # edges

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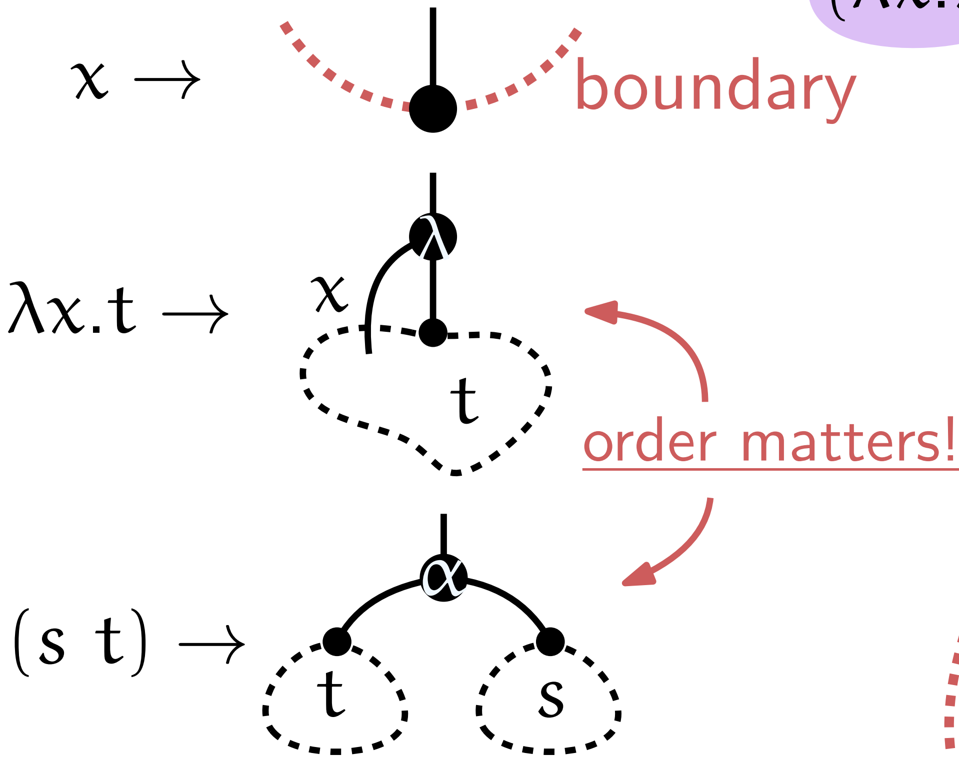

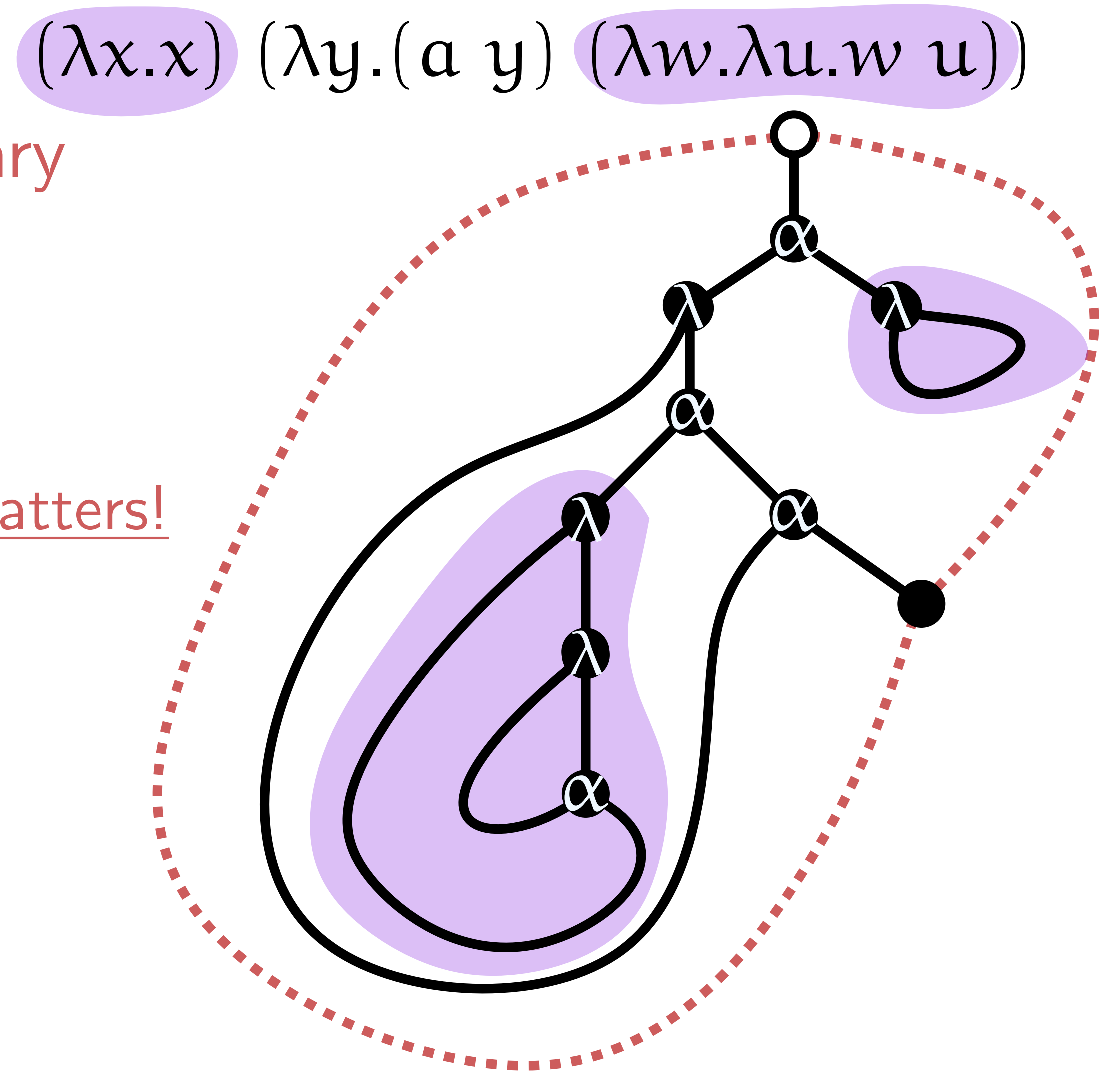

Dictionary

- $\#$ subterms $\leftrightarrow$ $\#$ edges
- closed subterms $\leftrightarrow$ bridges

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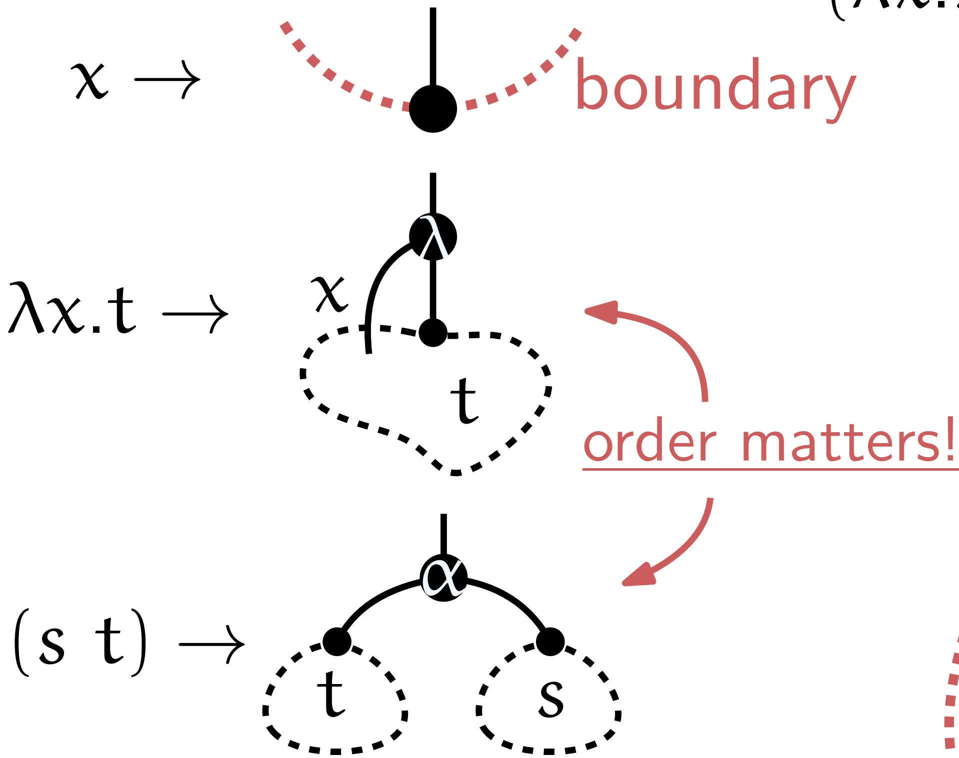

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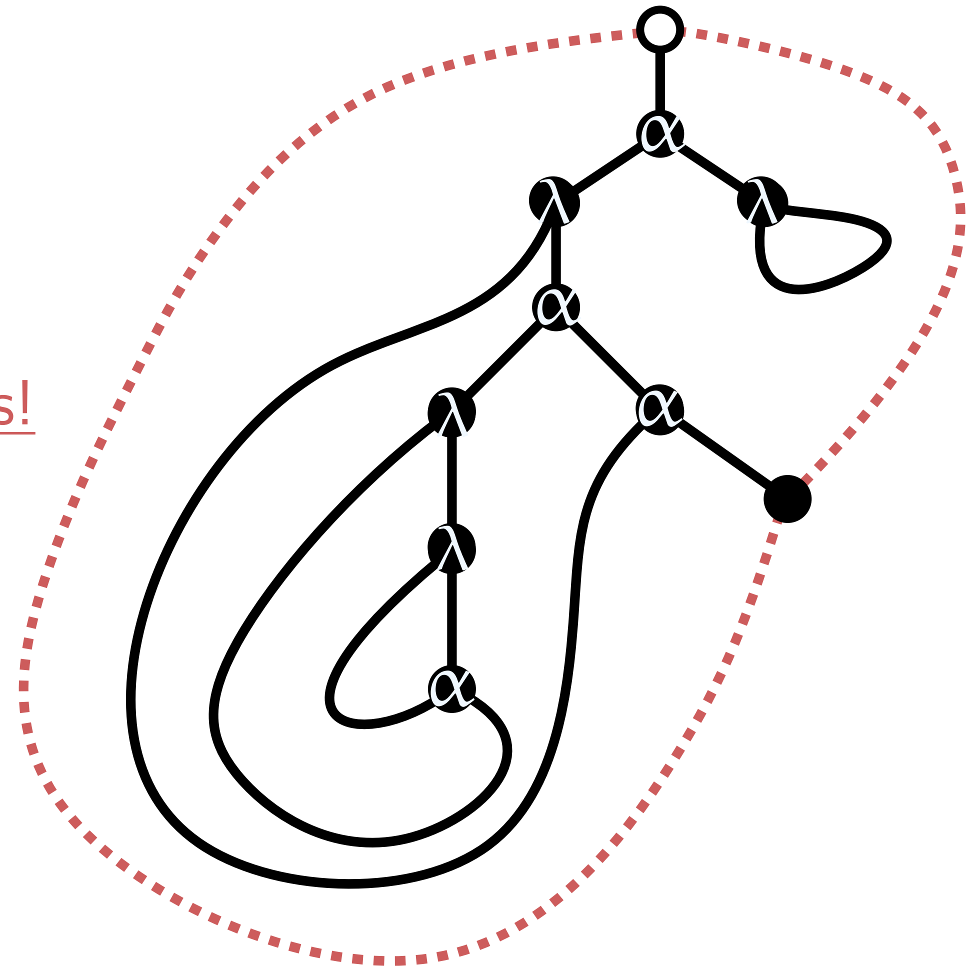

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- # subterms ↔ # edges
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- using variables in order ↔ planarity of diagrams

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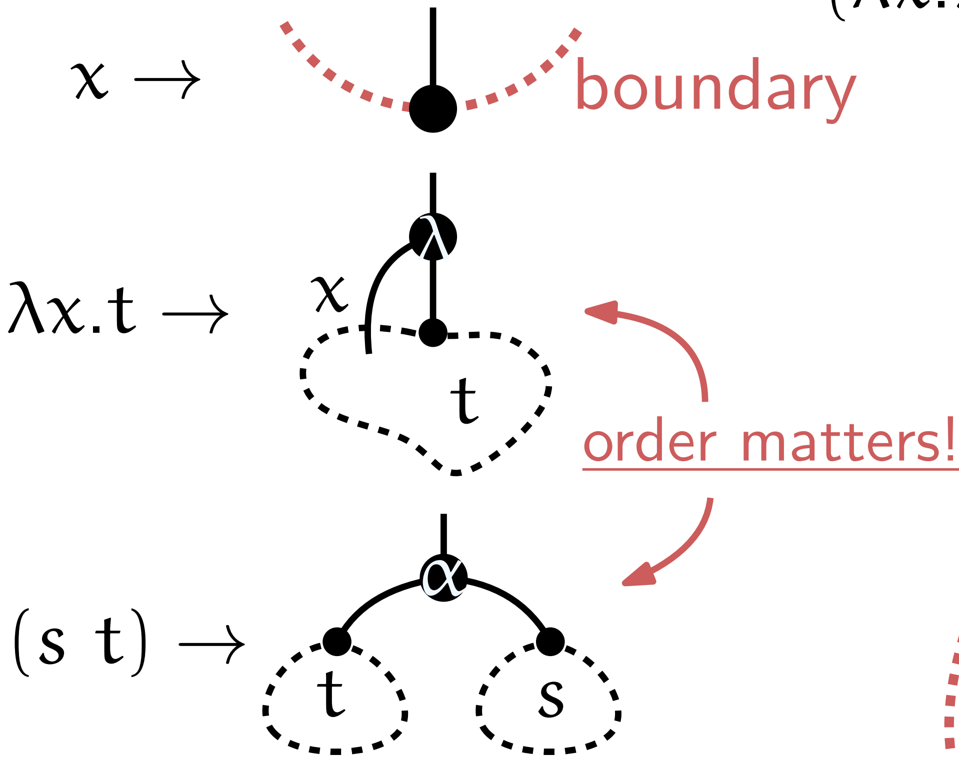

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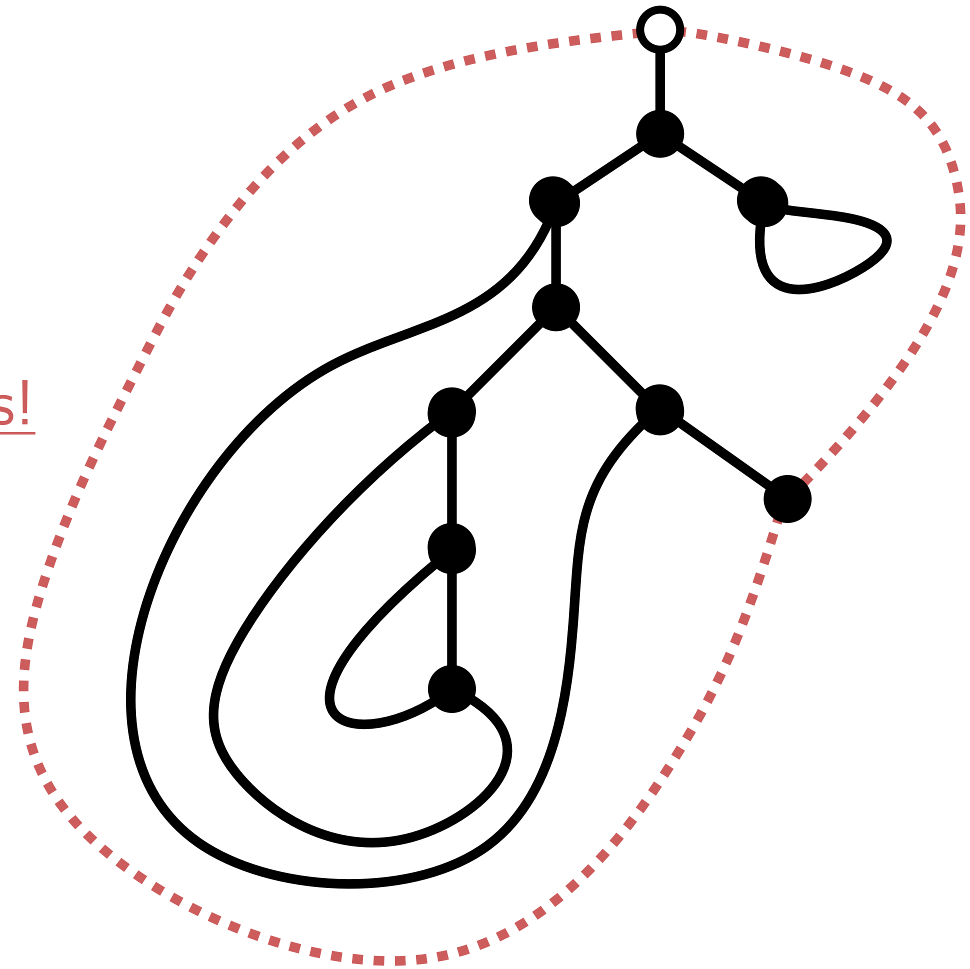

- # subterms $\leftrightarrow$ # edges
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Q: What if we erase the labels? Can we recover them?

A: Yes, via an exploration process! [BGJ13, ZG14]

Decomposing planar trivalent maps

(with a boundary)

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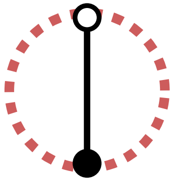

$$P(z, u) = uz$$

Decomposing planar trivalent maps

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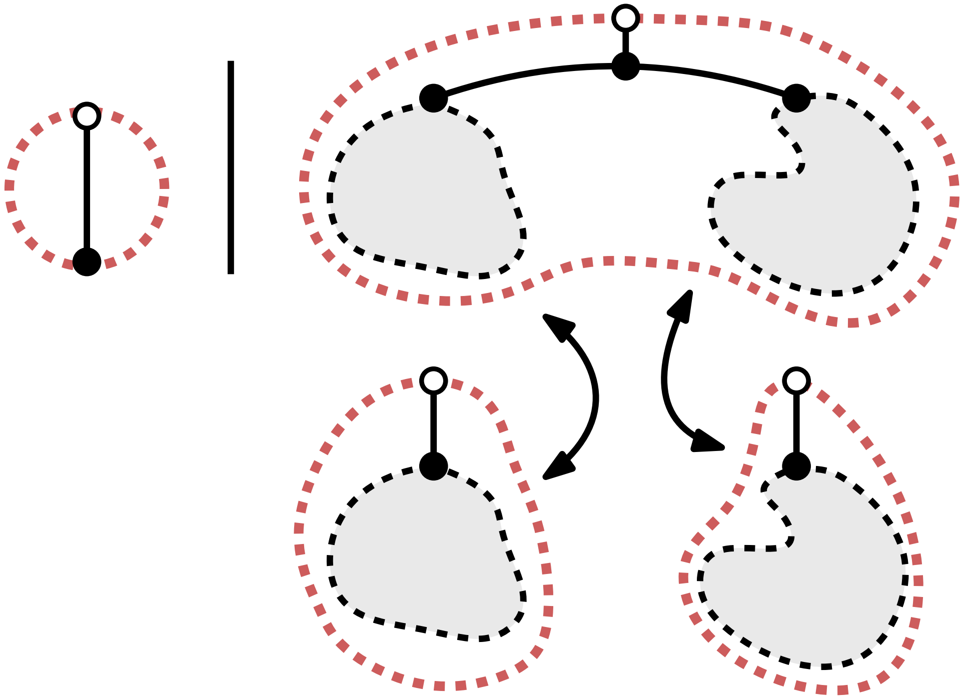

edges

non-root unary vertices

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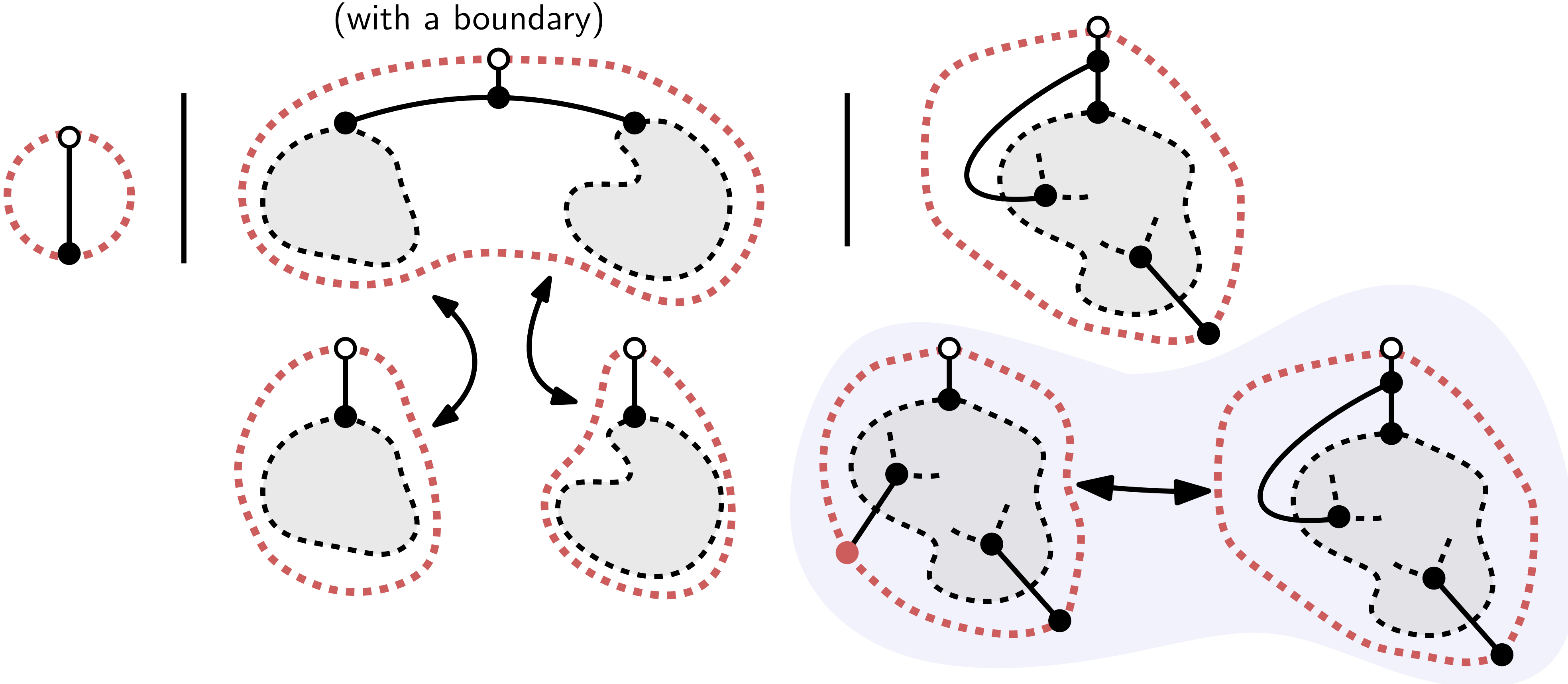

edges

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$$P(z,u) = uz + zP(z,u)^2 + z\frac{P(z,u) - P(z,0)}{u}$$

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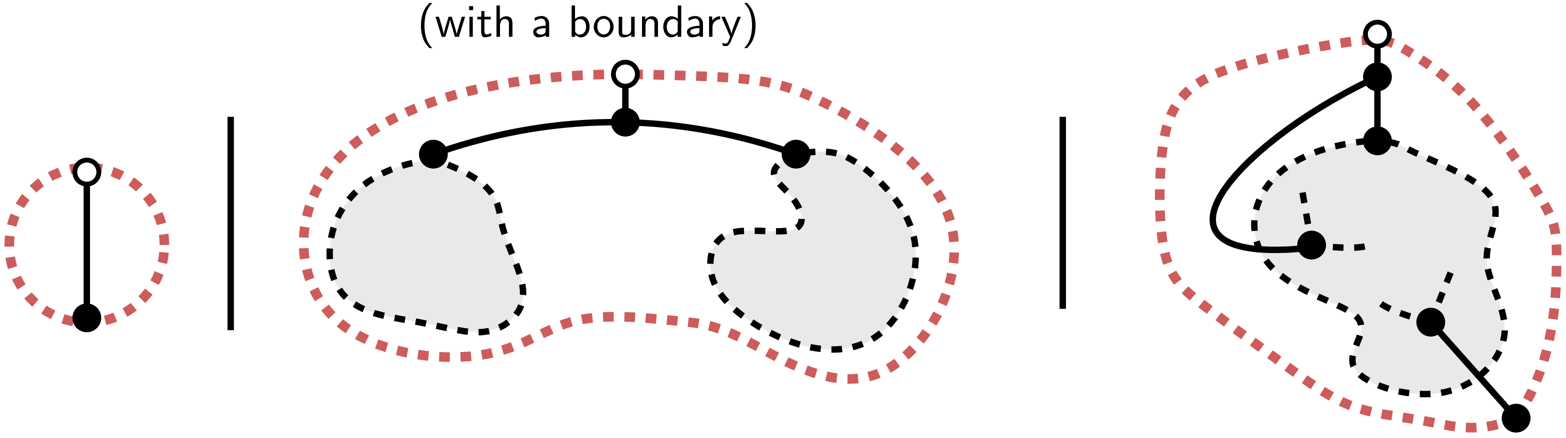

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Context of at least 1 var
&
Consume rightmost one

edges

non-root unary vertices

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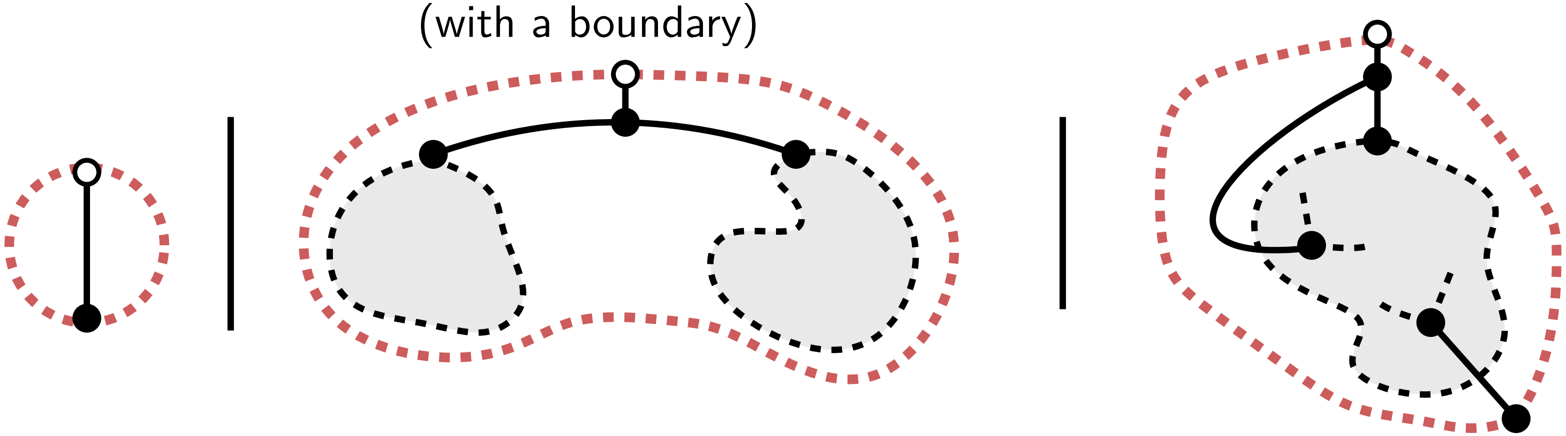
Decomposing planar trivalent maps and open planar terms!

(with a boundary)



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free vars.

subterms

Decomposing planar trivalent maps and open planar terms!

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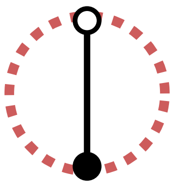
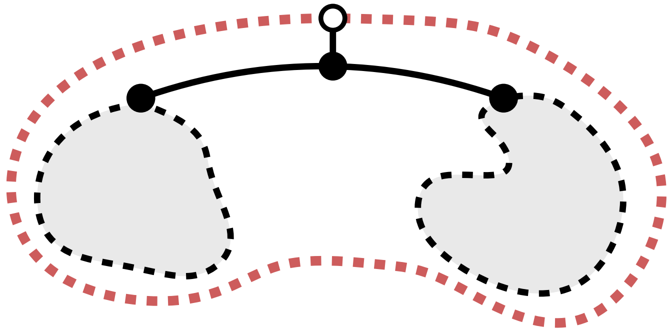
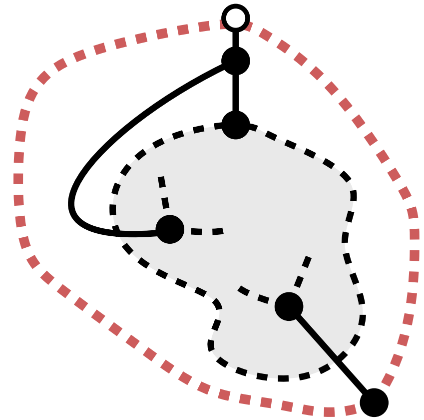
 |  | 

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For arbitrary genus replace $z\frac{F(z,u)-F(z,0)}{u}$ by $z\partial_u F(z,u)$!

edges

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$$P(z,u) = uz + zP(z,u)^2 + z\frac{P(z,u)-P(z,0)}{u}$$

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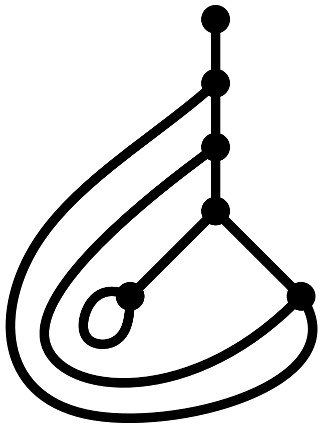
Closed planar terms and contexts

- Restricting the previous bijection we have:

  closed planar terms $\Leftrightarrow$ rooted trivalent planar maps

$\lambda x.\lambda y.((x\ y)\ (\lambda z.z)) \quad \leftrightarrow$ 

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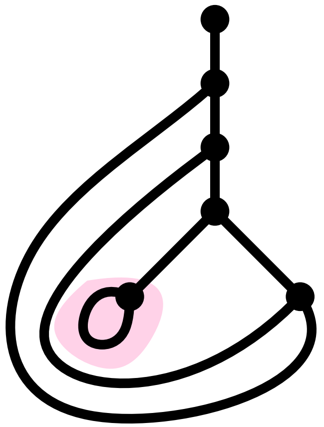
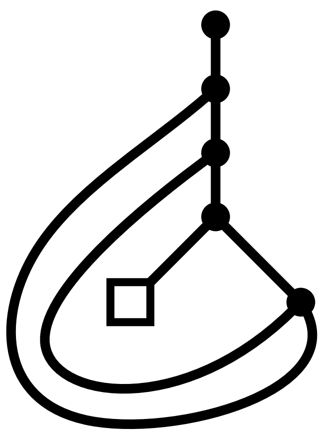


- We can also consider contexts:

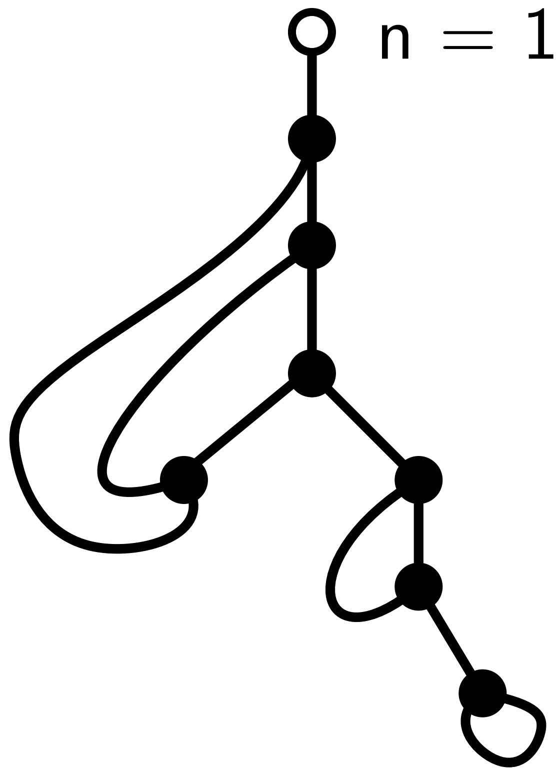
$$\lambda x.\lambda y.((x\ y)\ \square) \quad \leftrightarrow$$



Closed planar terms and contexts

Lemma

A closed planar term with $n = 3k + 2, k \in \mathbb{N}$, subterms has:



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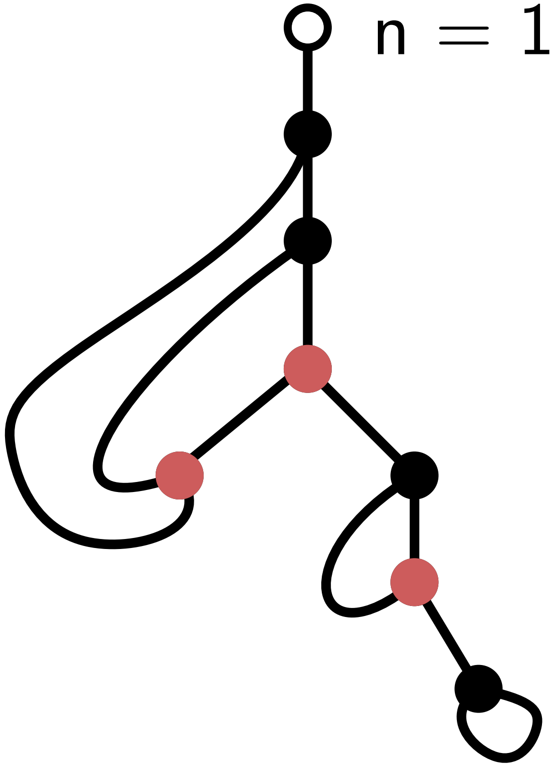

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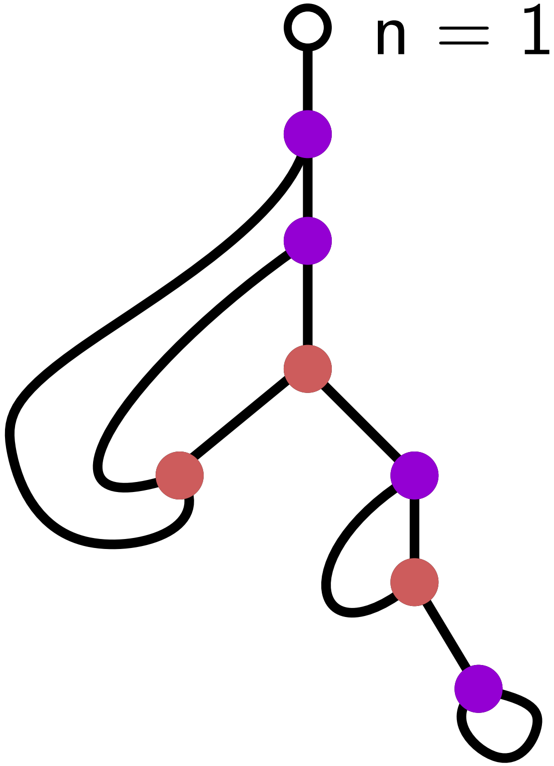

Closed planar terms and contexts

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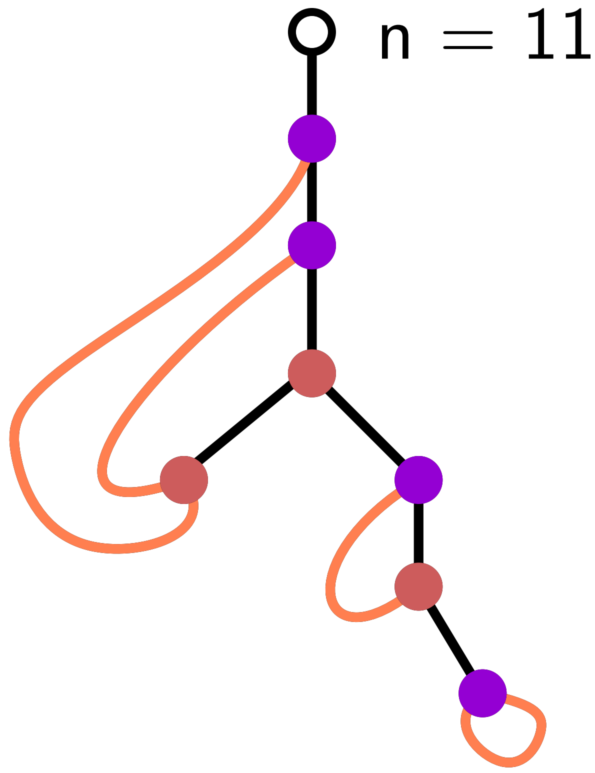
- k applications
- $k + 1$ abstractions
- $k + 1$ variables



The planar Goulden-Jackson recurrence

In [GJ08], Goulden and Jackson give the following recurrence for $F(k,g) = \#$ of rooted triangulations of k faces and genus g:

$$F(k,g) = \frac{f(k,g)}{3k+2}, \text{ for } (k,g) \in S \setminus \{(-1,0),(0,0)\},$$

where $S = \{(k,g) \in \mathbb{Z}^2 \mid k \geqslant -1, 0 \leqslant g \leqslant \frac{k+1}{2}\}$ and $f(k,g)$ is

$$f(-1,0) = \frac{1}{2}$$

$$f(k,g) = 0, \text{ for } (k,g) \notin S.$$

$$f(k,g) = \frac{4(3k+2)}{k+1}\left(k(3k-2)f(k-2,g-1) + \sum f(i,h)f(j,\ell)\right),$$

with the sum being taken over all pairs $(i,h) \in S$, $(j,\ell) \in S$ such that $i+j = k-2$ and $h+\ell = g$.

The planar Goulden-Jackson recurrence

using the KP hierarchy!

In [GJ08], Goulden and Jackson give the following recurrence for $F(k,g) = \#$ of rooted triangulations of k faces and genus g:

$$F(k,g) = \frac{f(k,g)}{3k+2}, \ \text{for } (k,g) \in S \setminus \{(-1,0),(0,0)\},$$

where $S = \{(k,g) \in \mathbb{Z}^2 \mid k \geqslant -1, 0 \leqslant g \leqslant \frac{k+1}{2}\}$ and $f(k,g)$ is

$$f(-1,0) = \frac{1}{2}$$

$$f(k,g) = 0, \ \text{for } (k,g) \notin S.$$

$$f(k,g) = \frac{4(3k+2)}{k+1}\left(k(3k-2)f(k-2,g-1) + \sum f(i,h)f(j,\ell)\right),$$

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Open problem: give a combinatorial interpretation of the above. Planar case resolved by Baptiste Louf [B19].

The planar Goulden-Jackson recurrence

Reparameterising and setting $g = 0$, we have:

$$u(0) = 1$$

$$u(k+1) = 2(3k+2)p(k)$$

$$(k+1)p(k) = \sum_{i=0}^{n} u(i)u(n-i)$$

where $p(k)$ counts

The planar Goulden-Jackson recurrence

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- rooted planar triangulations with $2k$ faces
- rooted planar trivalent maps with $2k$ vertices

duality

Notice the *apparent* shift in size notion!

$3k+2$ edges $\leftrightarrow$ $2k$ vertices

The planar Goulden-Jackson recurrence

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- rooted planar triangulations with $2k$ faces

  ↓ duality

- rooted planar trivalent maps with $2k$ vertices

  ↓ bijection

- closed planar terms with k applications

Notice the *apparent* shift in size notion!

$3k+2$ edges $\leftrightarrow$ $2k$ vertices

$3k+2$ subterms $\leftrightarrow$ k applications

How to (re)prove the Planar G&J Recurrence

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k applications $\Rightarrow 3k+2$ subterms

2 ways to introduce a new application

$$\lambda x.\lambda y.(x\ y) \Leftrightarrow \begin{array}{c} \lambda x.\lambda y.(\Box\ (x\ y)) \\ \text{or} \\ \lambda x.\lambda y.((x\ y)\ \Box) \end{array}$$

So, $u(k)$ counts contexts with k apps!

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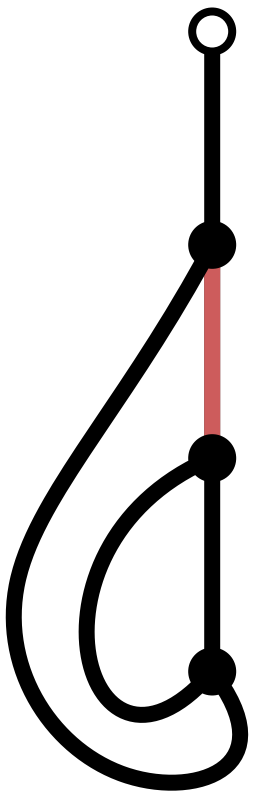

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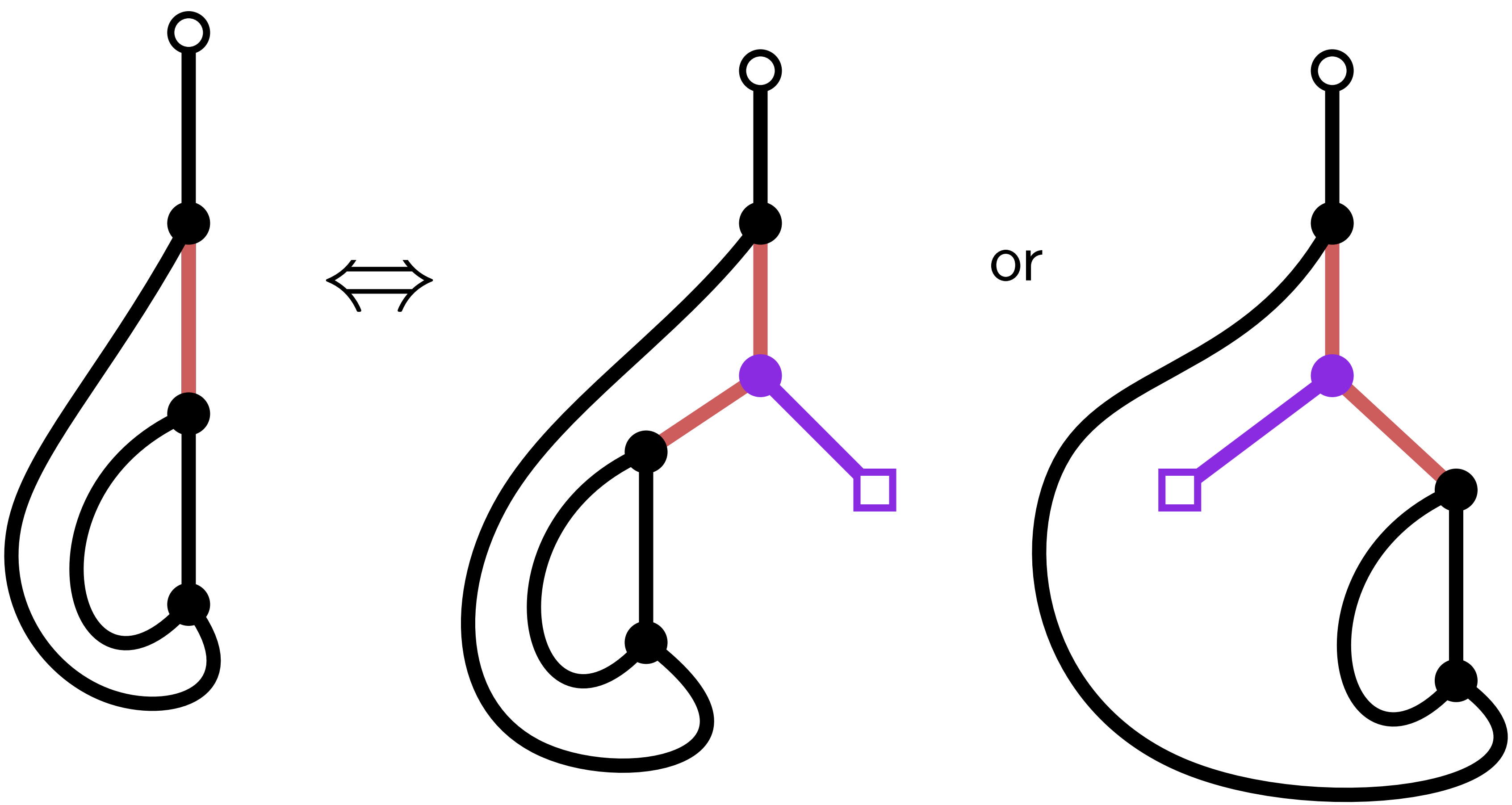

How to (re)prove the Planar G&J Recurrence

- Step 2:

$$(k+1)p(k) = \sum_{i=0}^{n} u(i)u(n-i)$$

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split var-pointed term into two contexts

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minimal closed subterm that contains v

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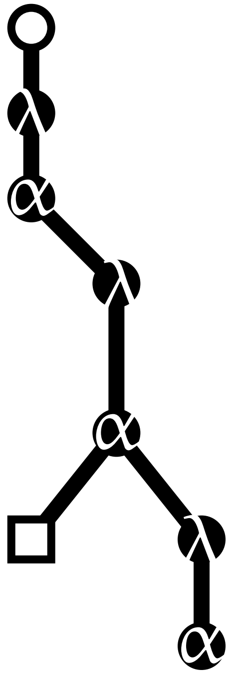
Lemma:



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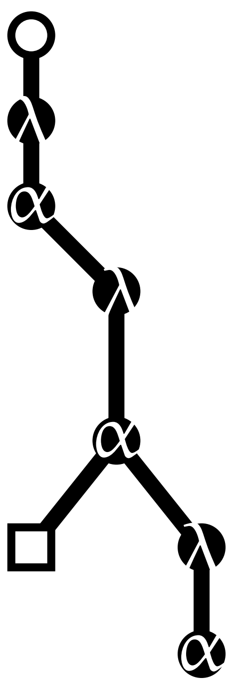
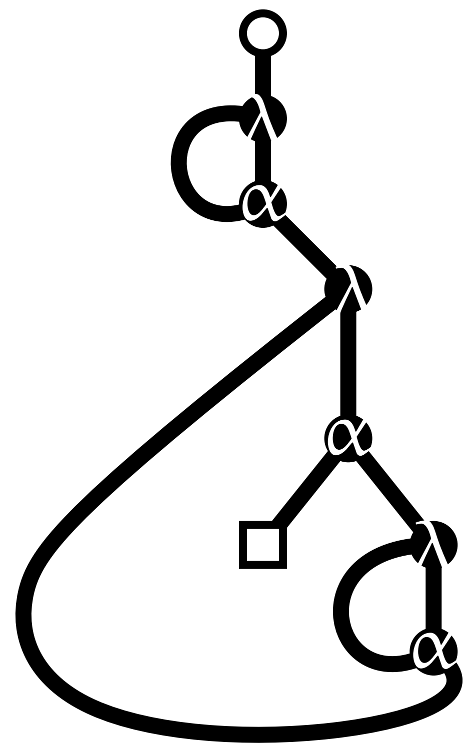


$\leftrightarrow$



$\lambda\text{␣}.(\lambda\text{␣}.((\lambda\text{␣}.\text{␣␣})\Box))\text{␣}$ $\quad\leftrightarrow\quad$ $\lambda x.(\lambda y.((\lambda z.yz)\Box))x$

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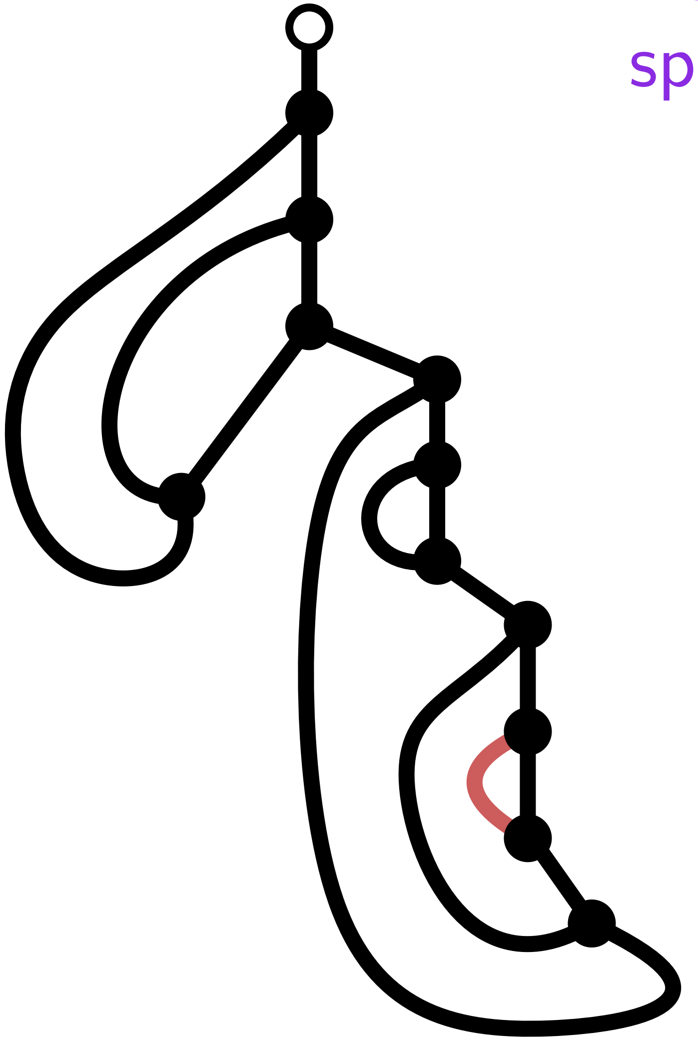

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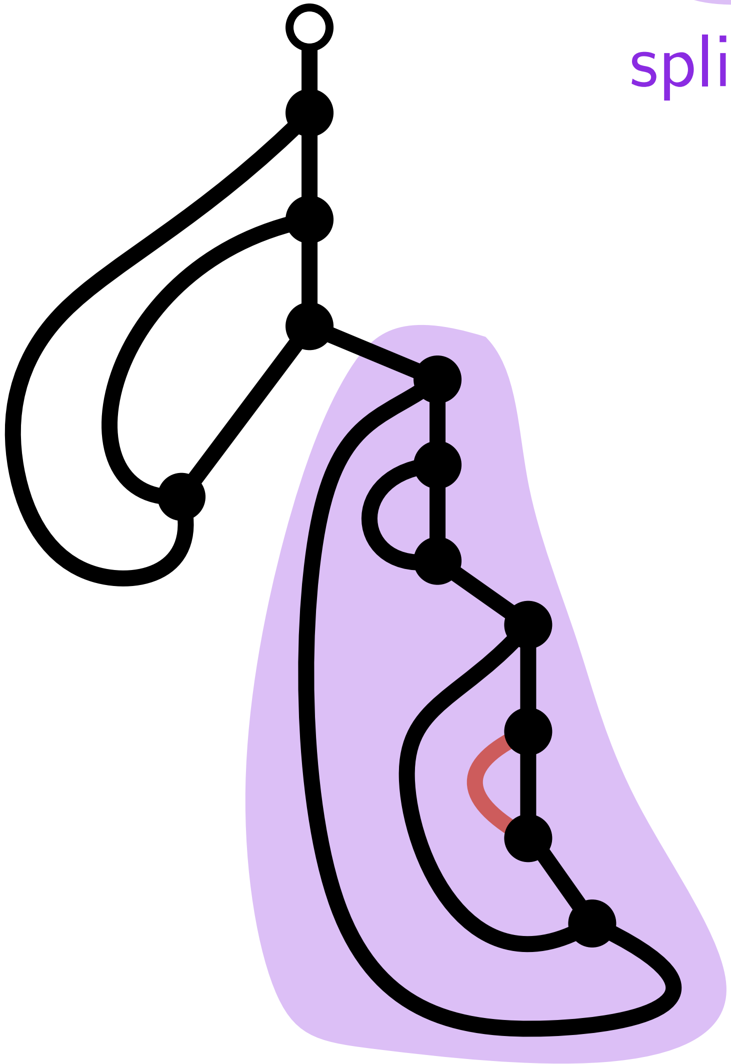

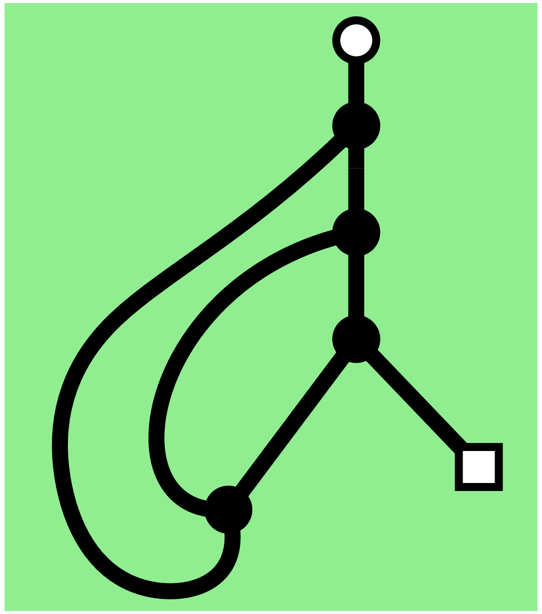

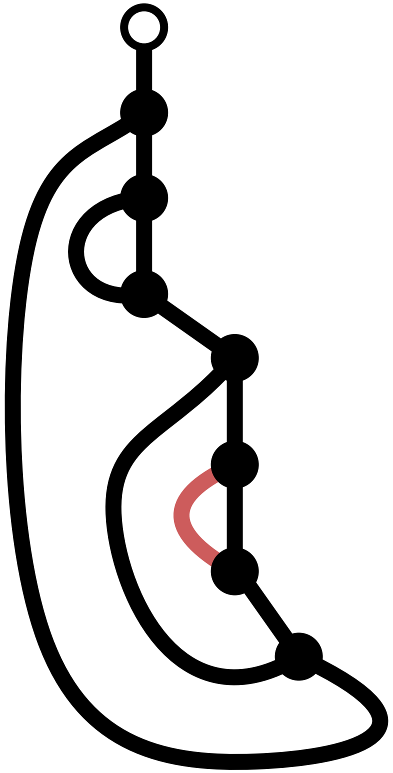

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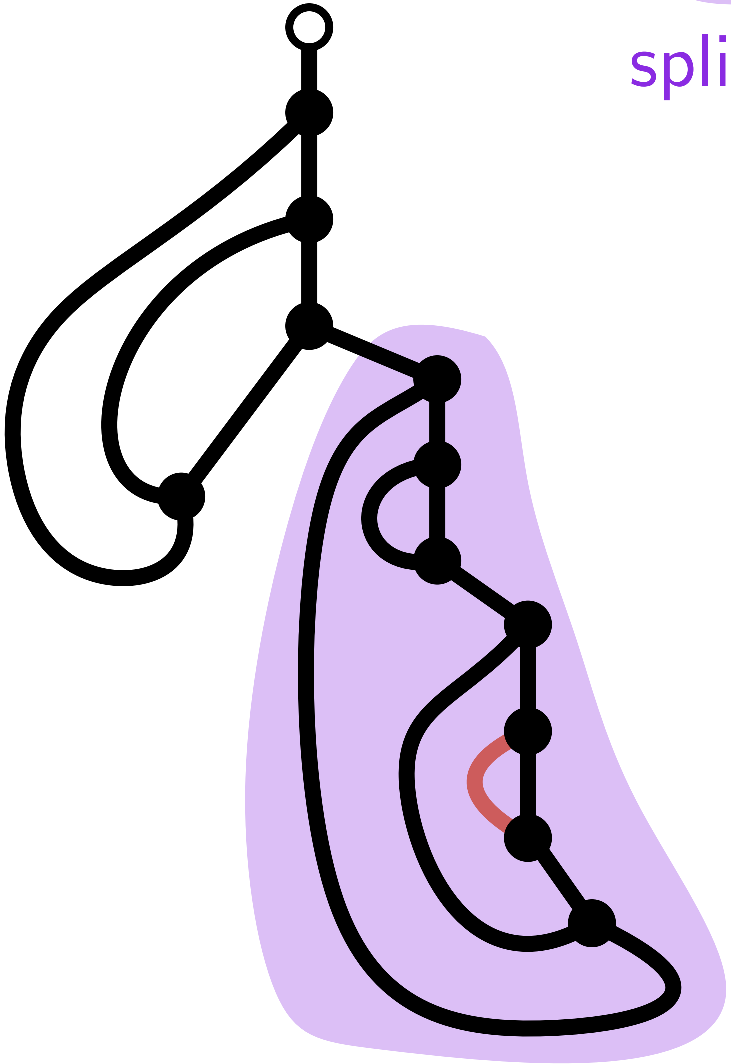

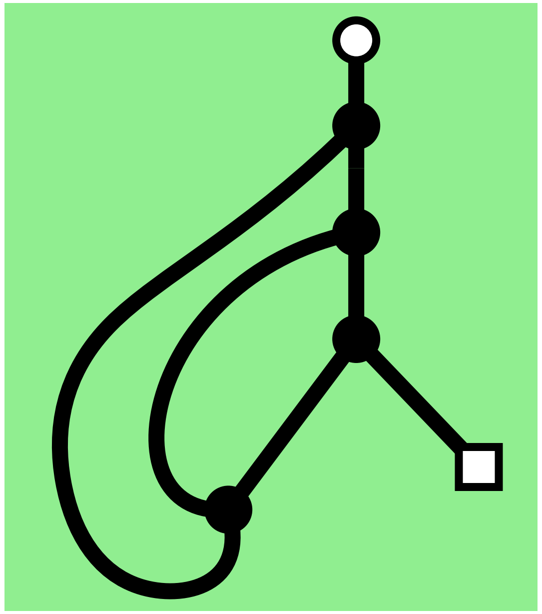

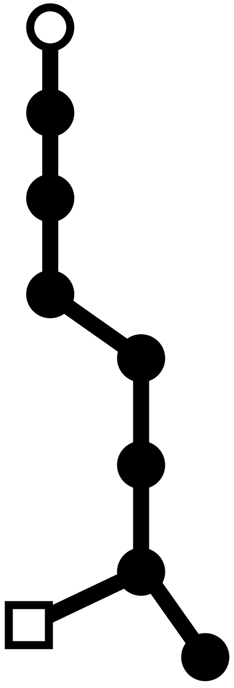

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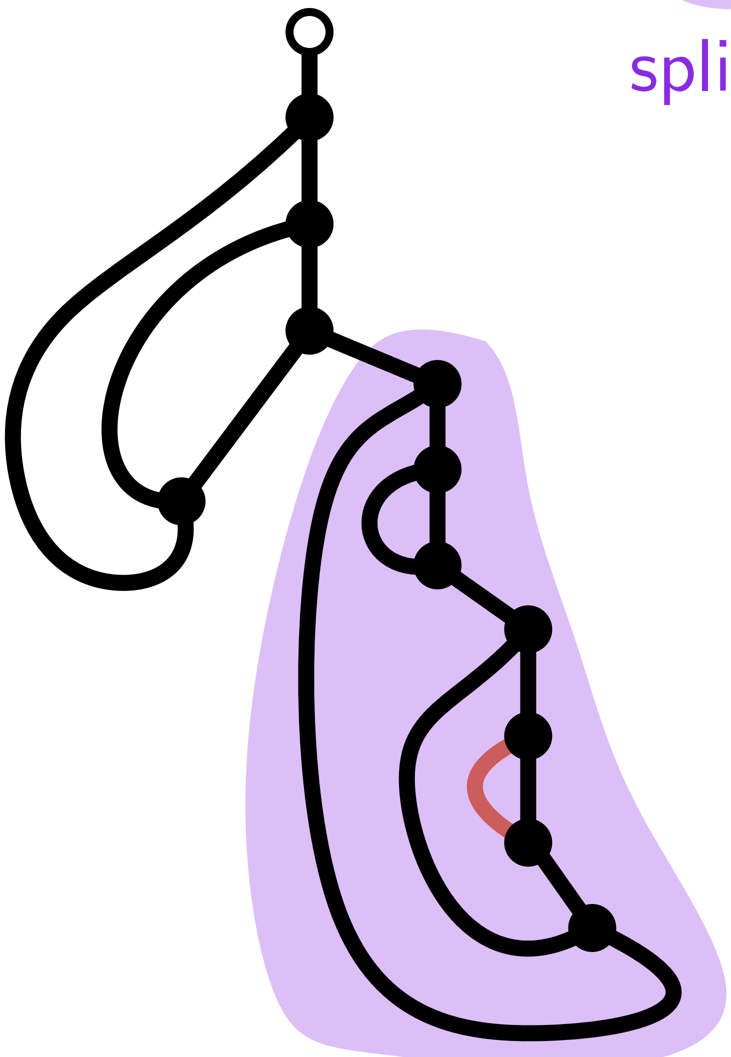

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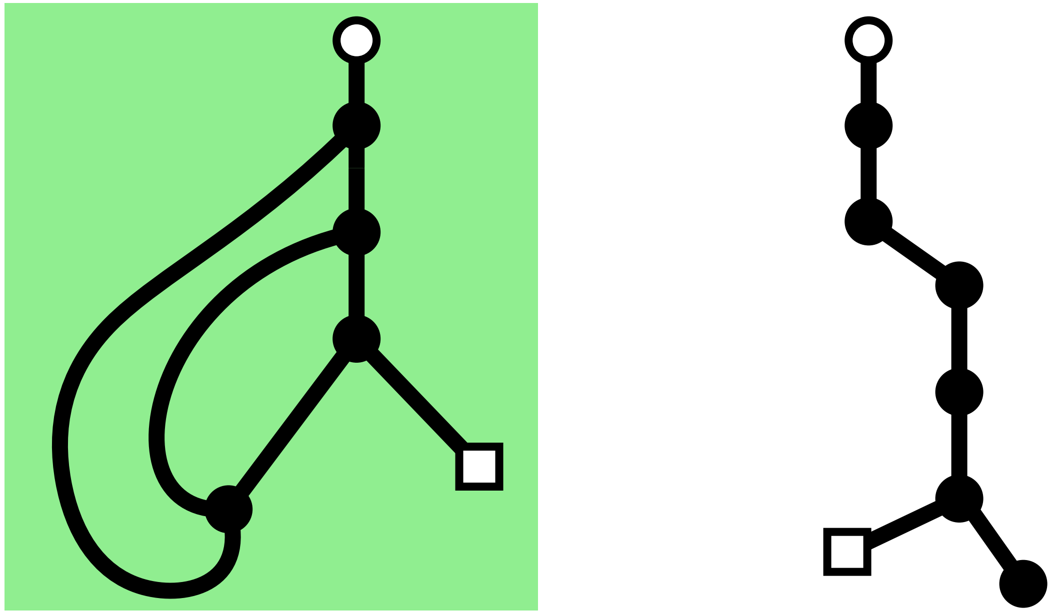

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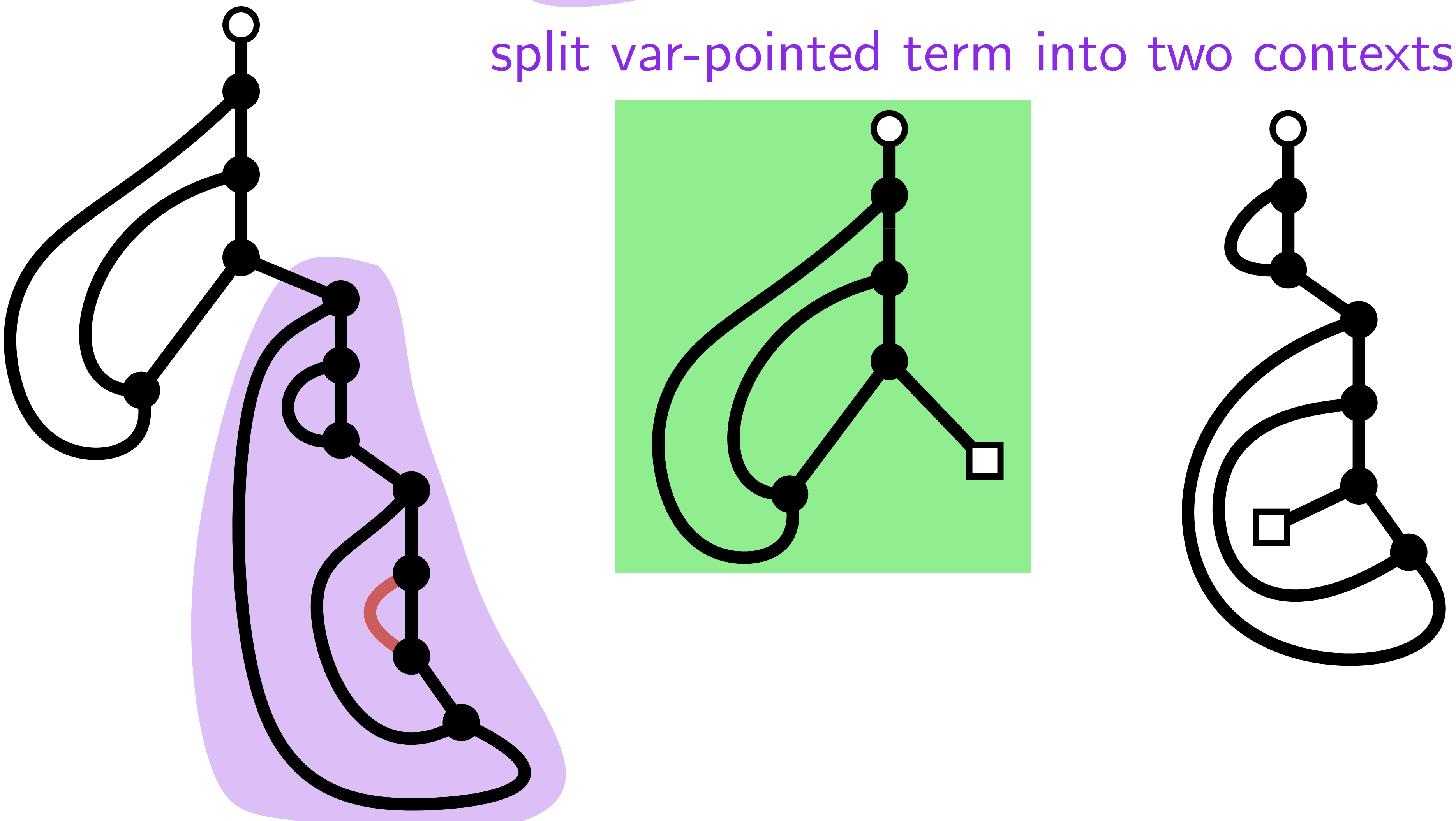

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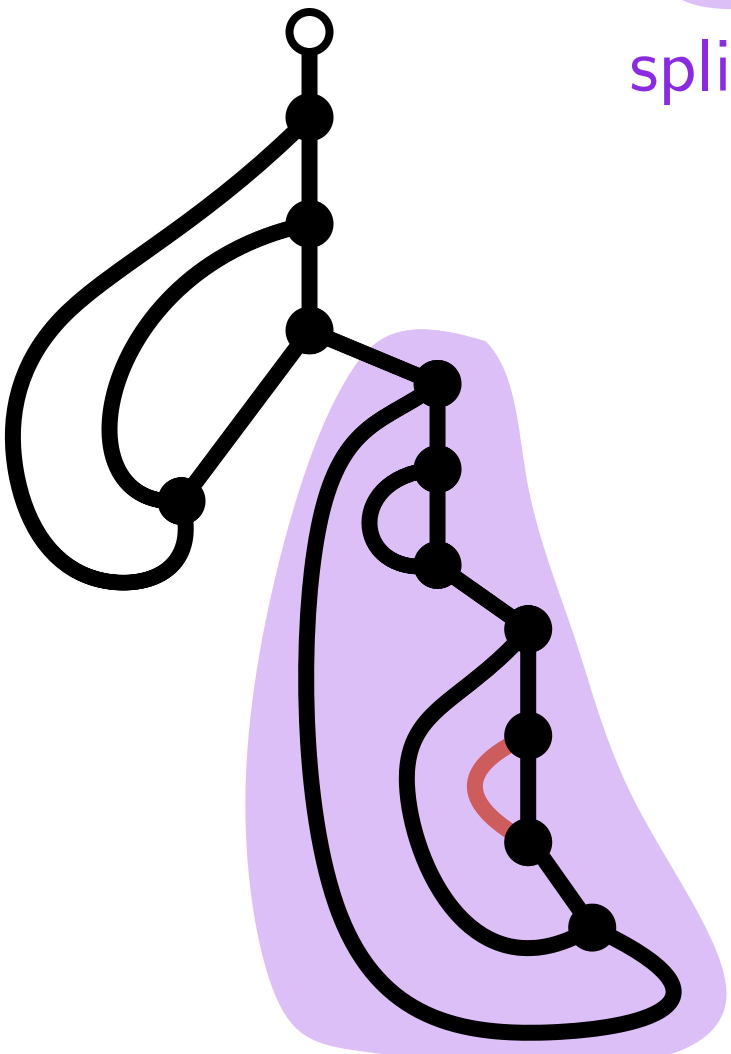

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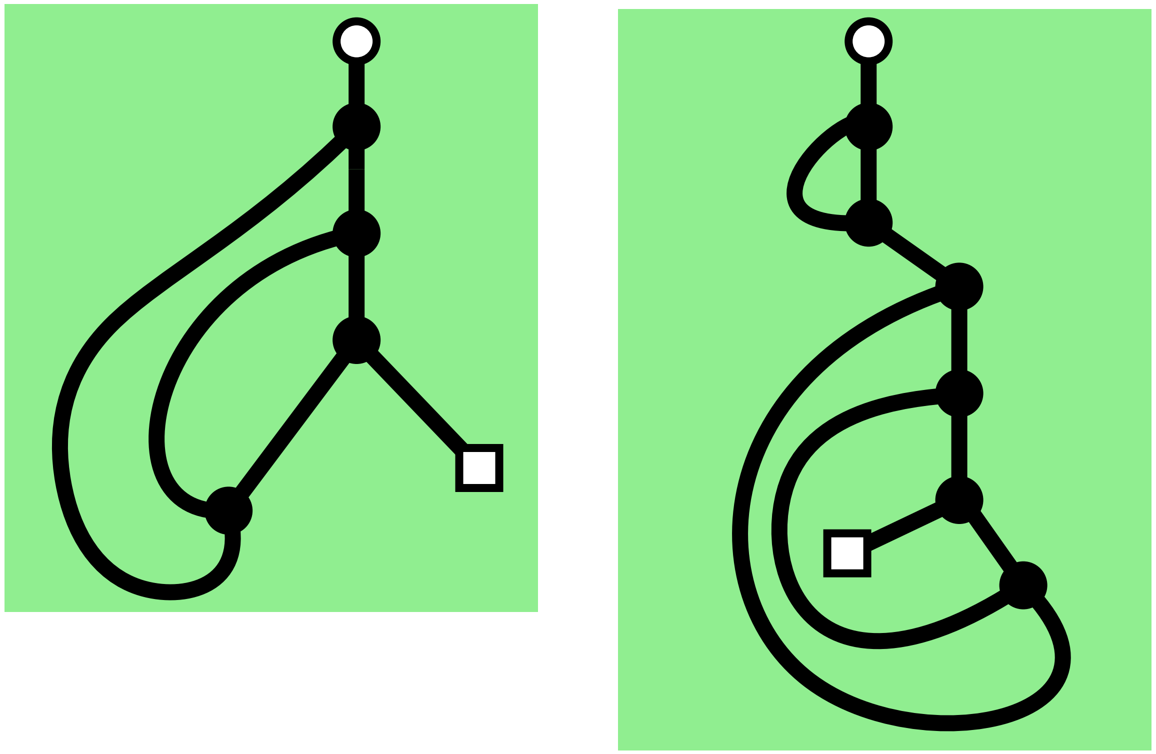

Some open problems

- Bijective interpretation of G&J rec. for general genus

$$o(0,g) = 1$$

$$o(k+1,g) = 2(3k+2)t(k,g)$$

$$(k+1)t(k,g) = 2k(3k-2)o(k-1,g-1) + \sum_{\substack{i+j=k \\ h+\ell=g}}^{n} o(i,h)o(j,\ell)$$

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- Genus for λ-terms?

Some open problems

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- Genus for λ-terms?

Thank you!

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