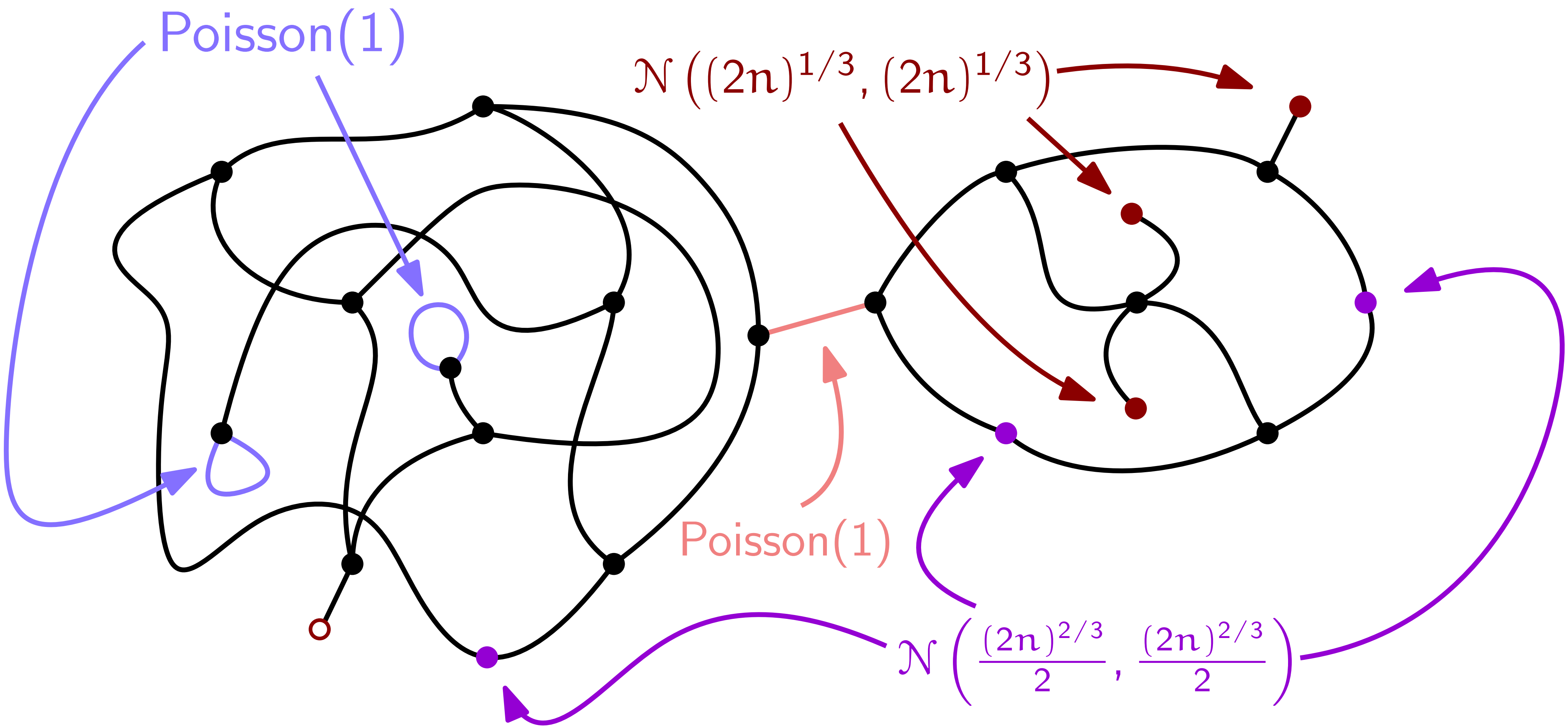
Distributions of parameters in restricted classes of maps and λ-terms



Séminaire AlGCo, 28 octobre 2021

Olivier Bodini (LIPN, Paris 13)
Alexandros Singh (LIPN, Paris 13)
Noam Zeilberger (LIX, Polytechnique)

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not in this talk!

Techniques drawn from combinatorics, logic, and physics may be used in tandem to study them!

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- We're interested in terms up to α-equivalence:

$$(\lambda x.xx)(\lambda x.xx) \overset{\alpha}{=} (\lambda y.yy)(\lambda x.xx) \overset{\alpha}{\neq} (\lambda y.ya)(\lambda x.xx)$$

Subfamilies of λ-terms

General terms: no restrictions on variable use

$\lambda x.\lambda y.x$

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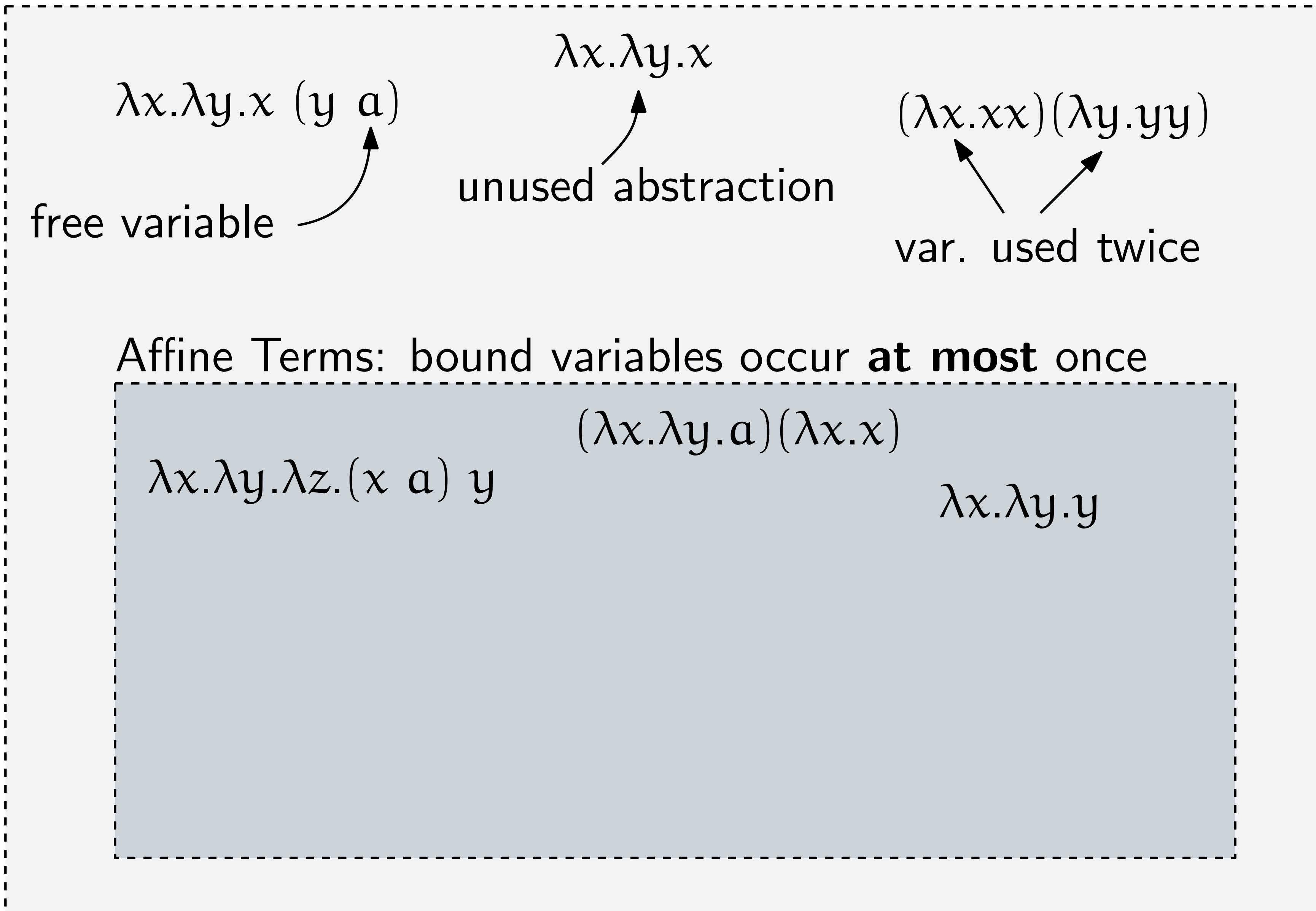

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Affine Terms: bound variables occur **at most** once

$(\lambda x.\lambda y.a)(\lambda x.x)$

$\lambda x.\lambda y.\lambda z.(x\ a)\ y$

$\lambda x.\lambda y.y$

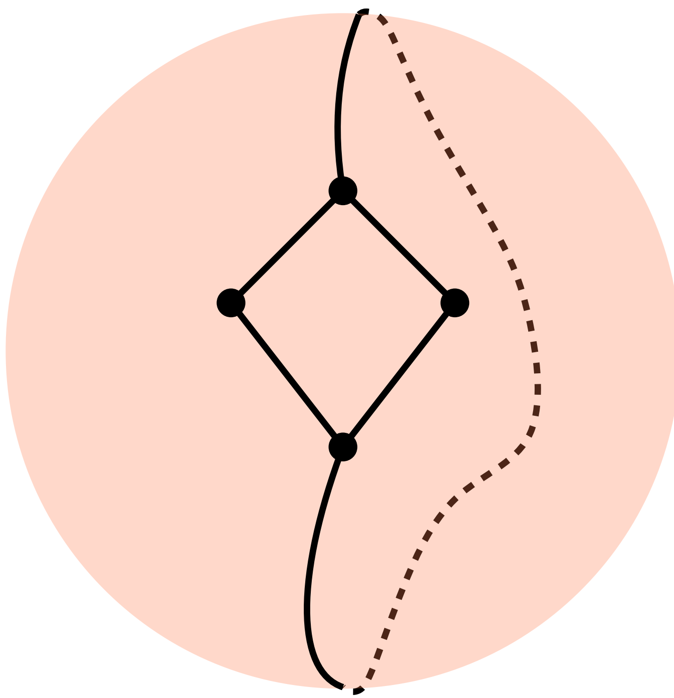
Linear Terms: bound variables occur **exactly** once

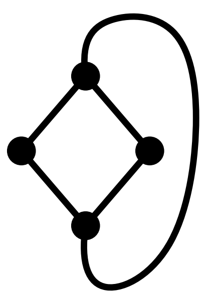
$\lambda x.\lambda y.(y\ x)a$

$\lambda x.\lambda y.(y\ a)(b\ x)$

$\lambda x.a(\lambda z.(\lambda y.y\ (x\ z)))$

What are maps?





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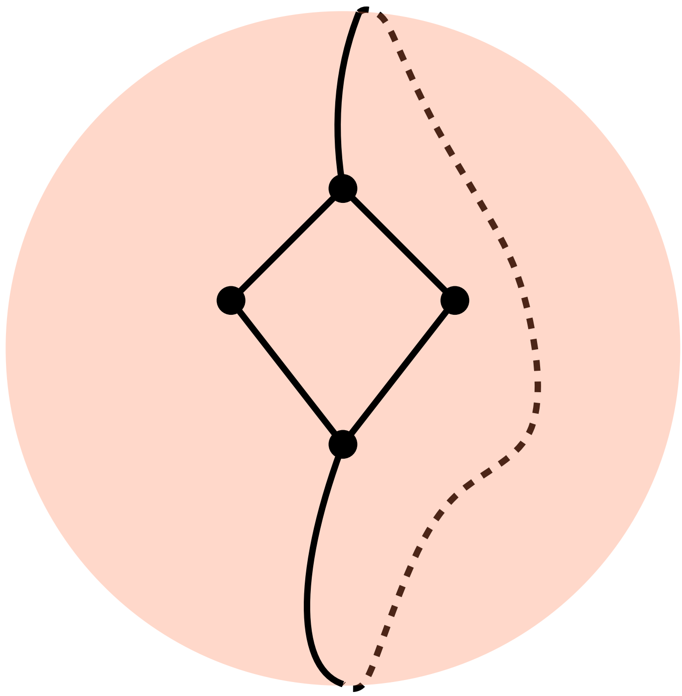

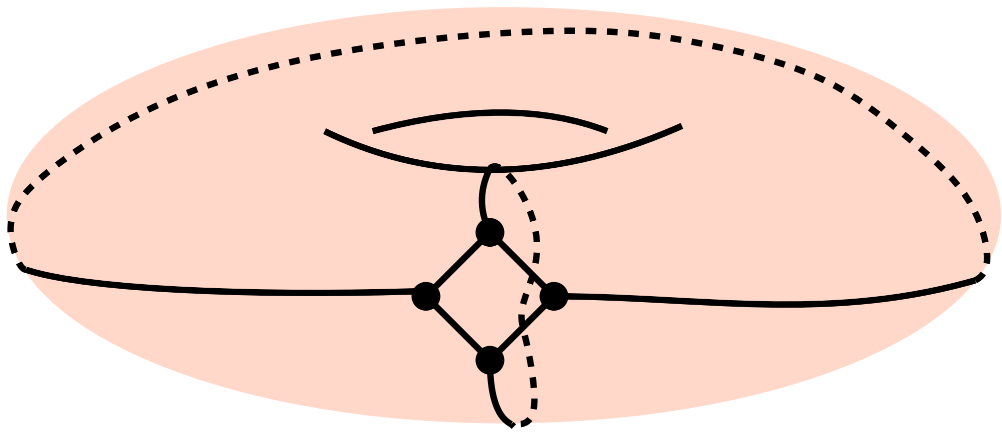

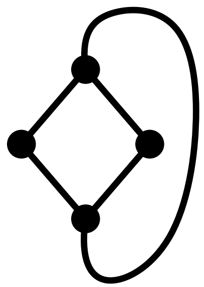

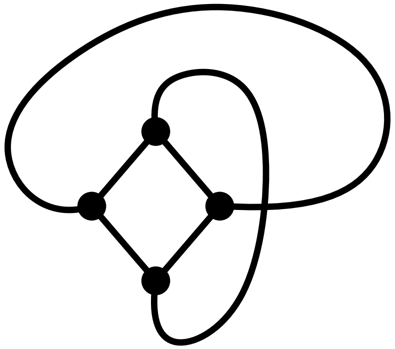

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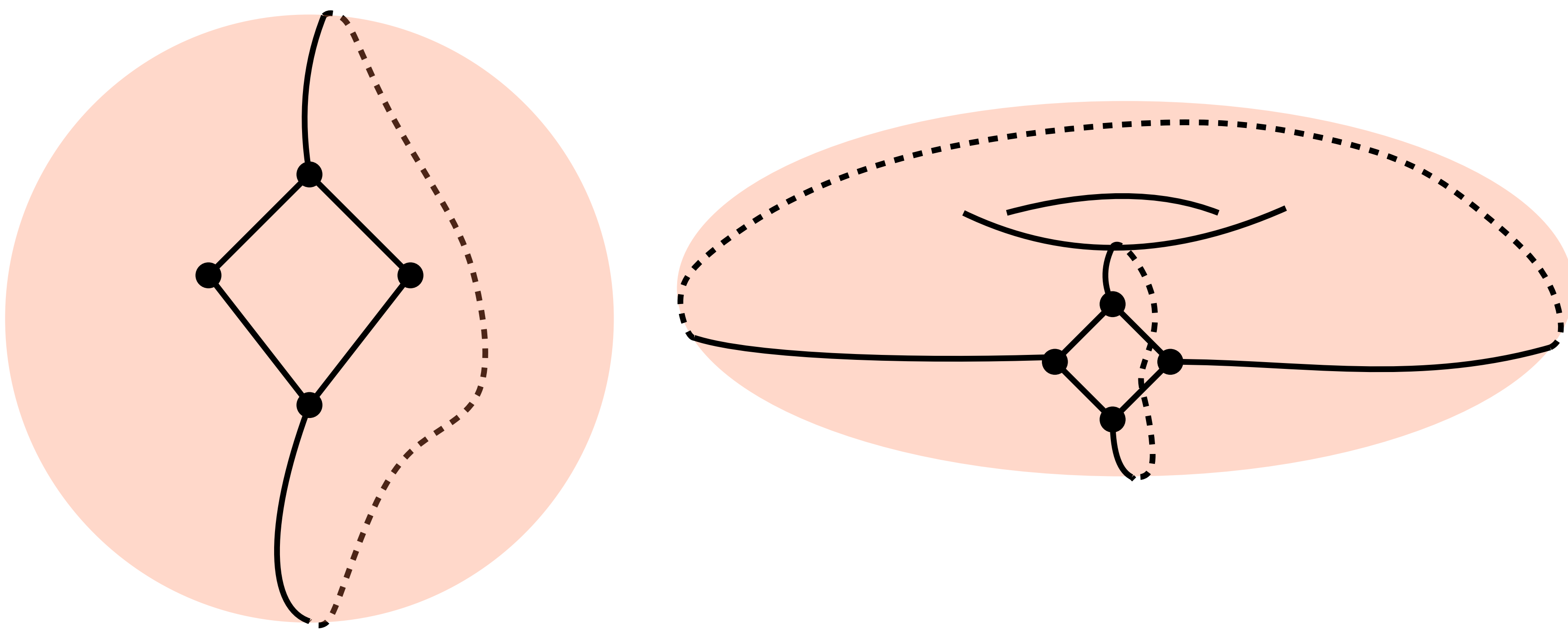
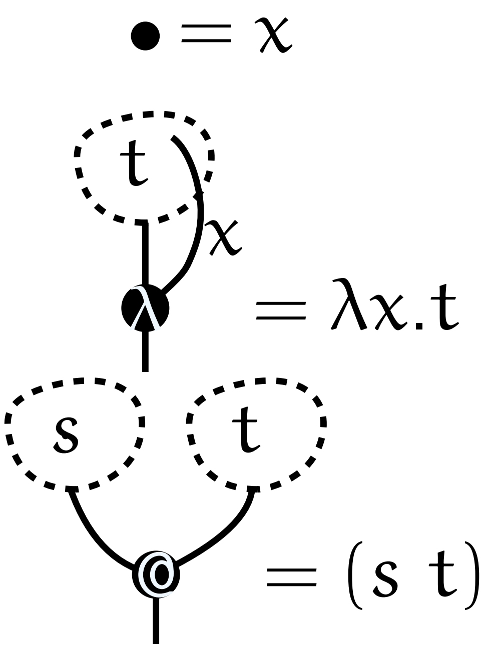


We're interested in unrestricted genus, restricted vertex degrees

Why should you, a logician, be interested in maps?

String diagrams! [BGJ13, Z16]



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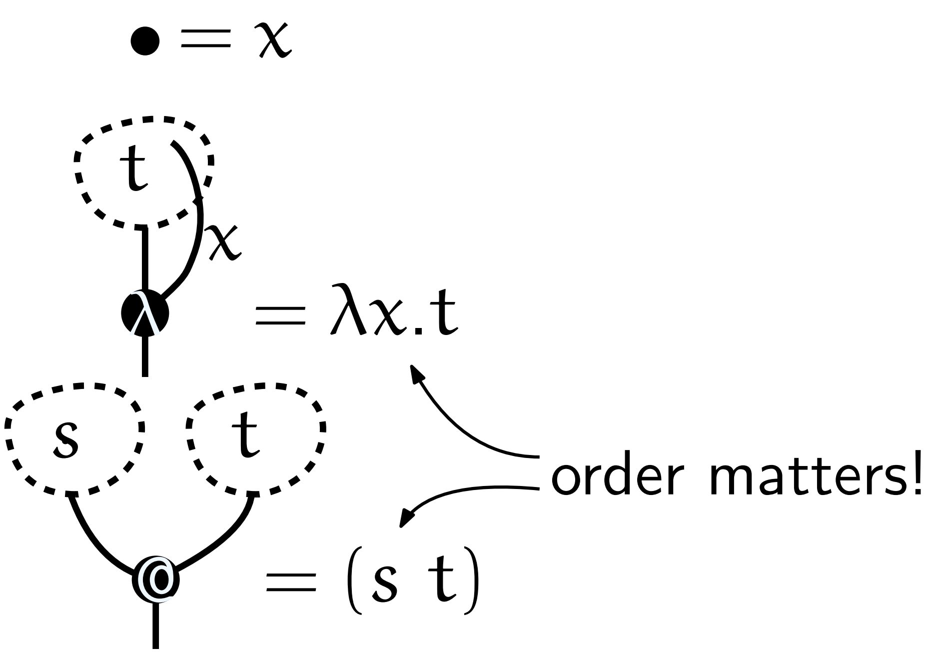

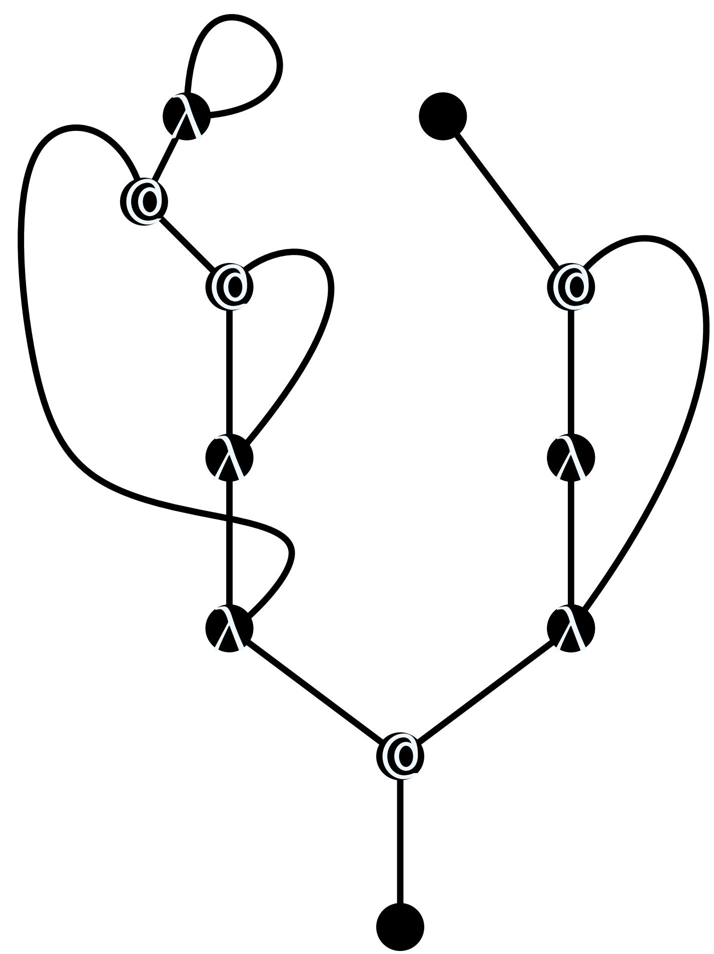

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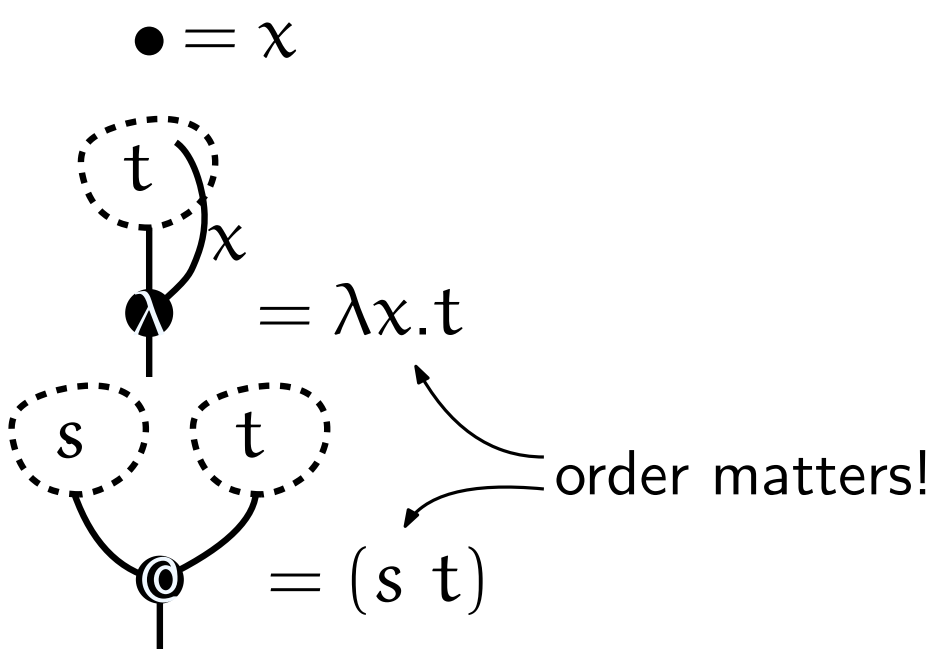

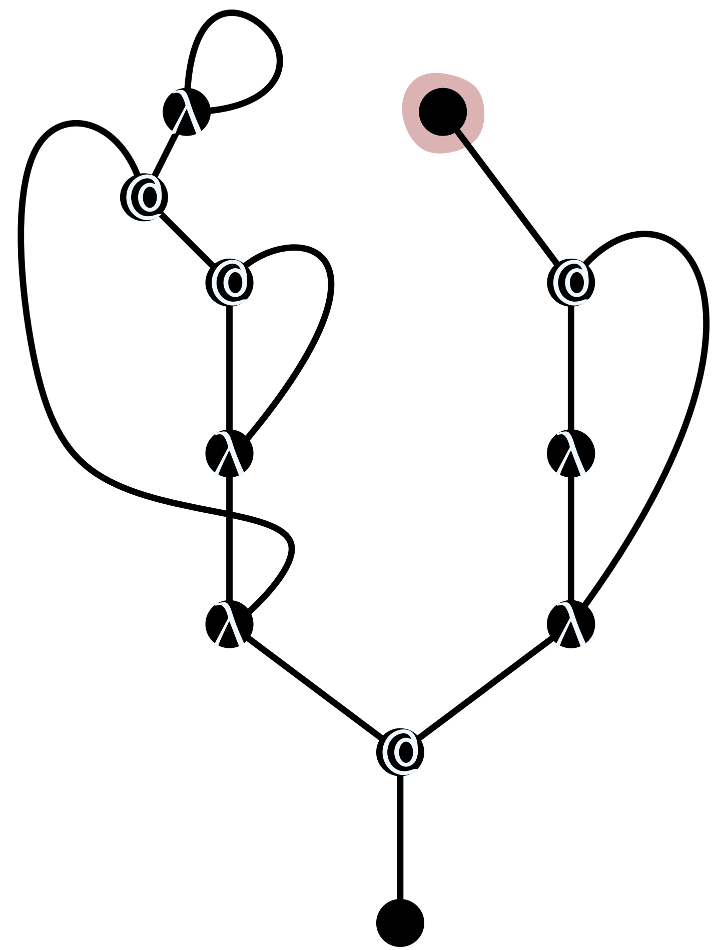

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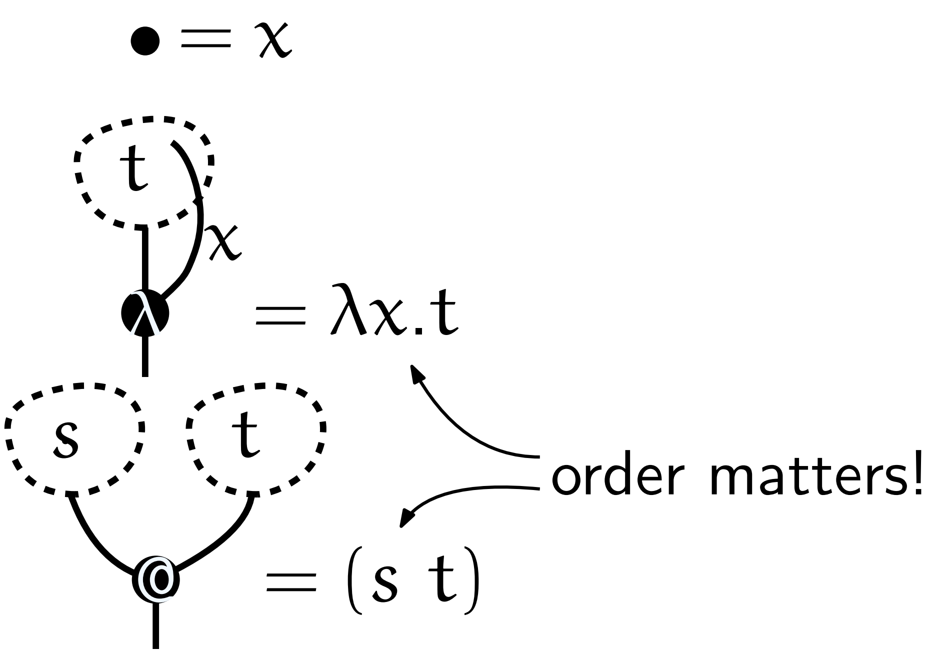

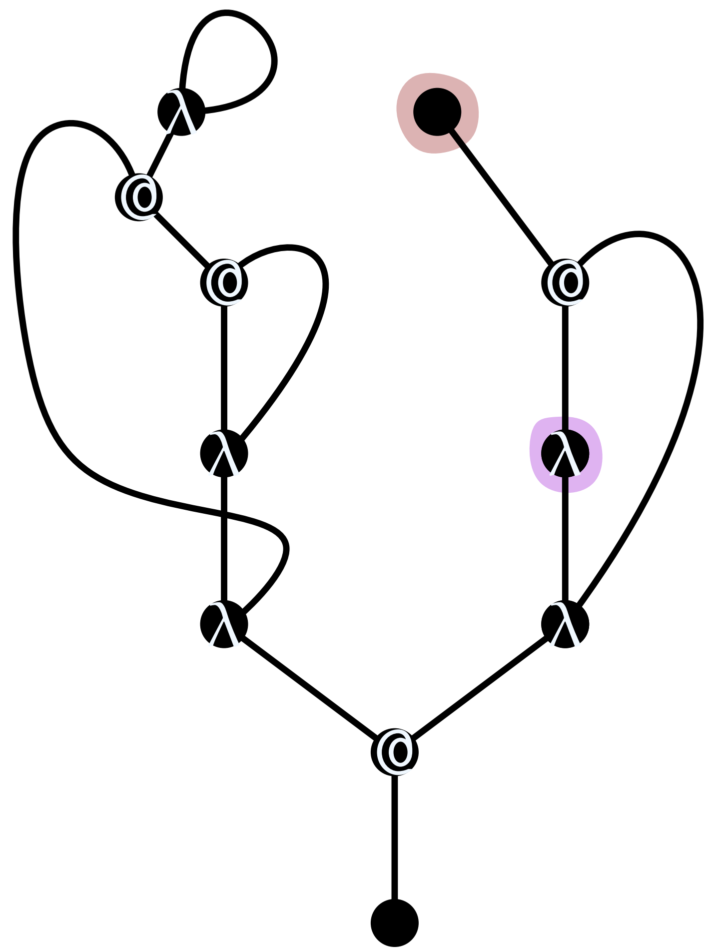

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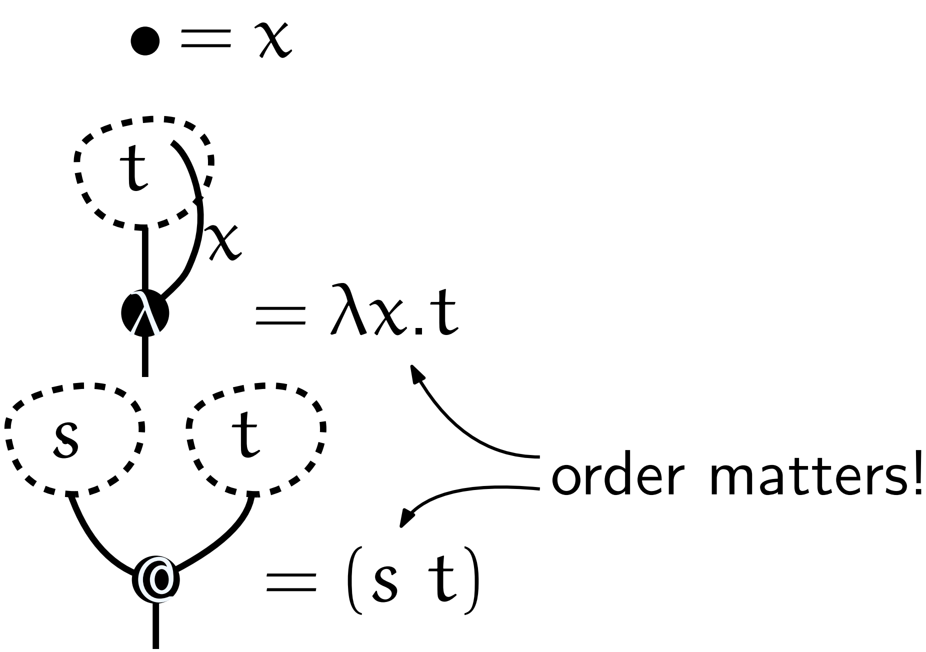

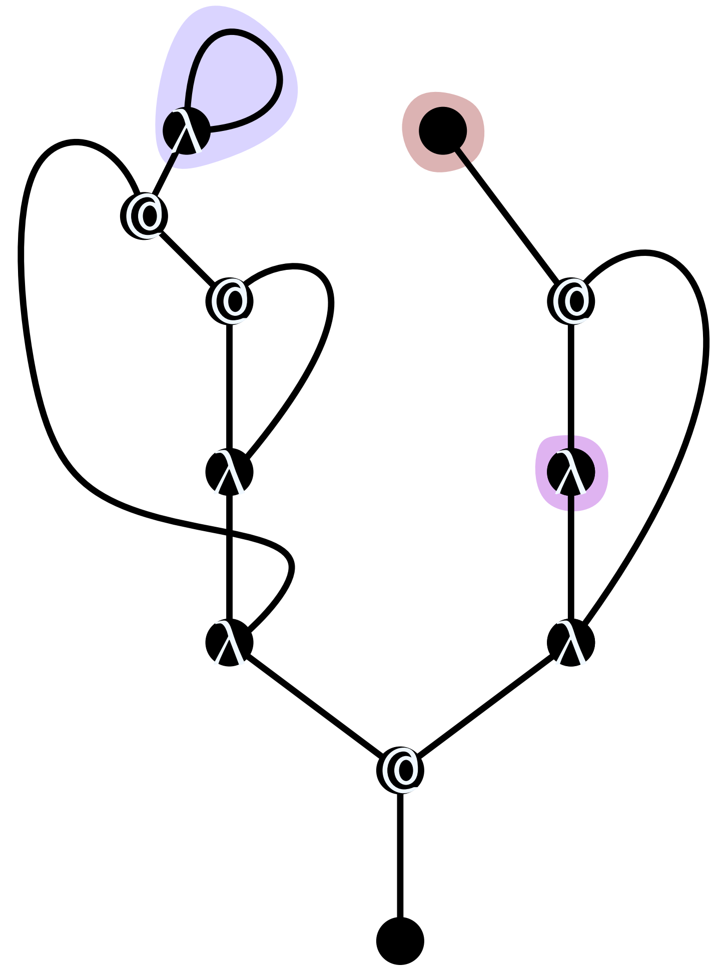

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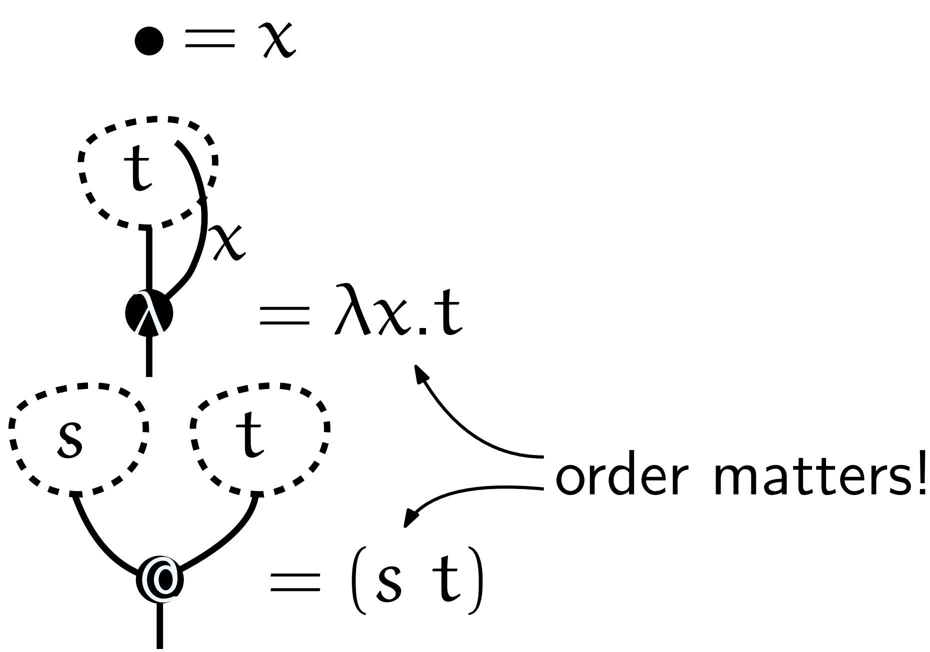

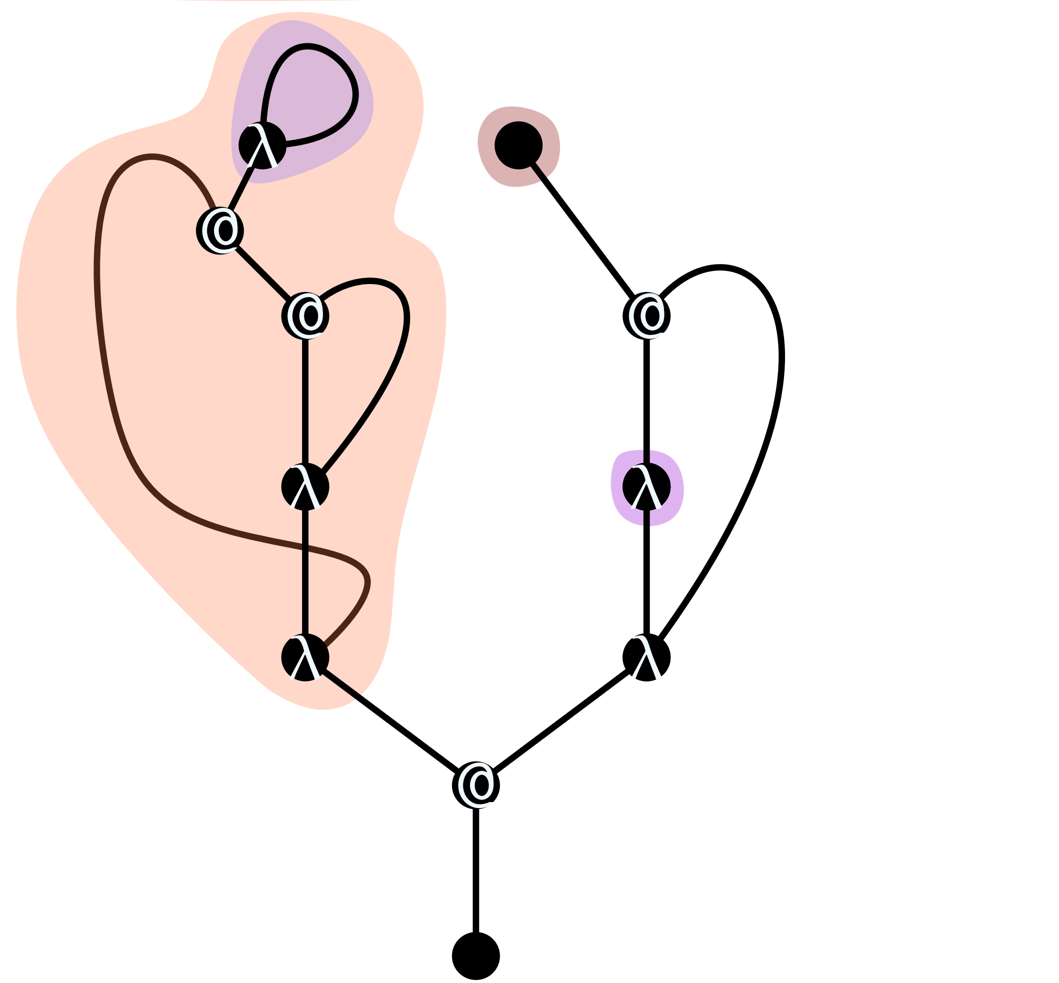

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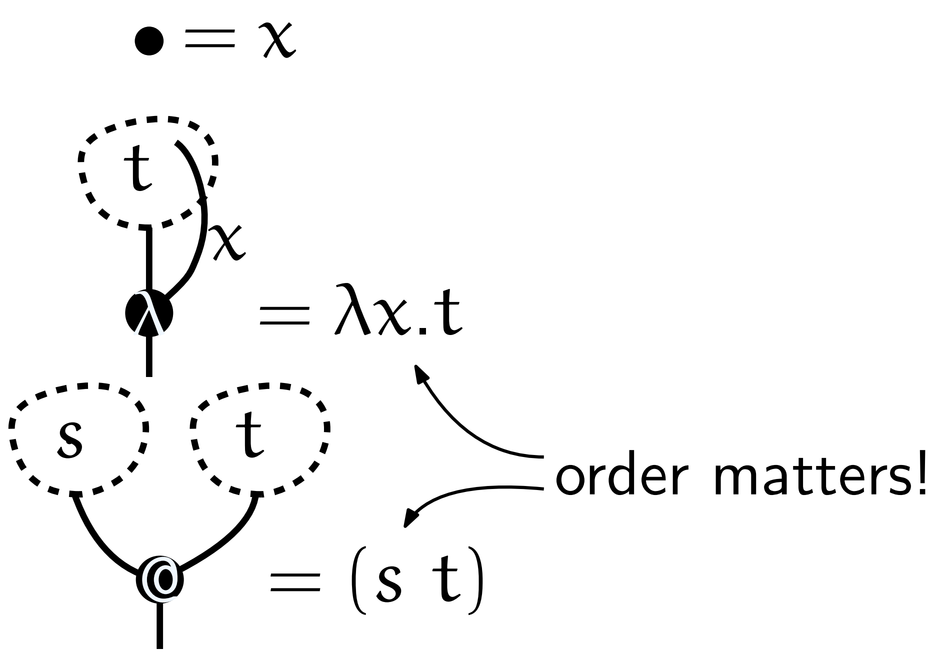

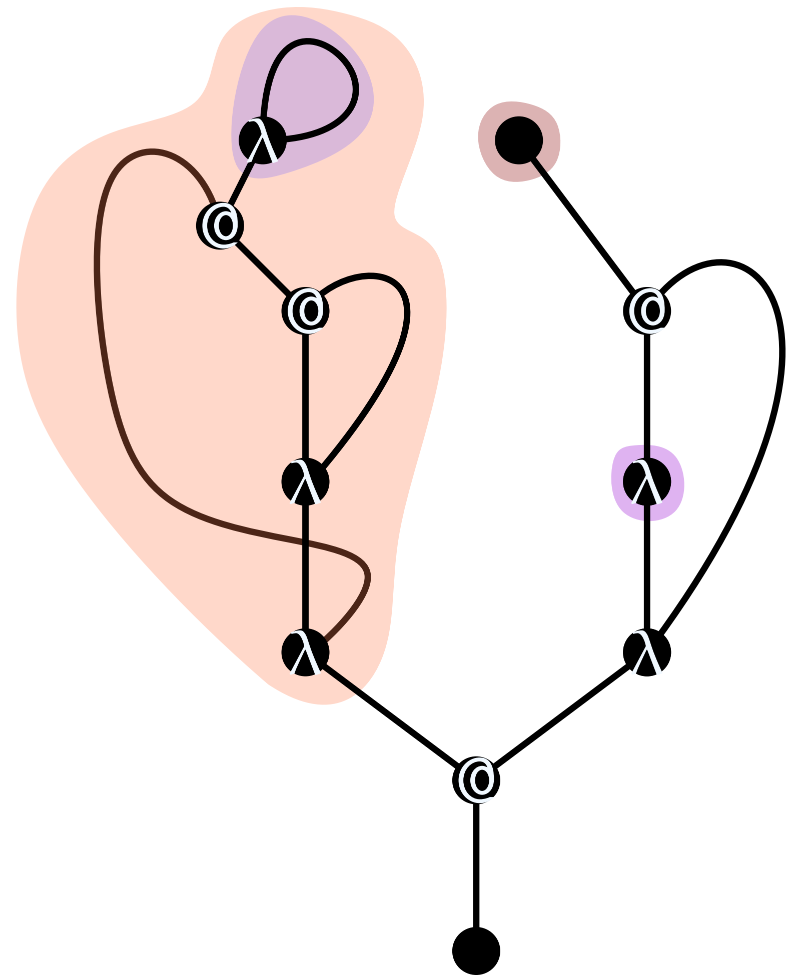

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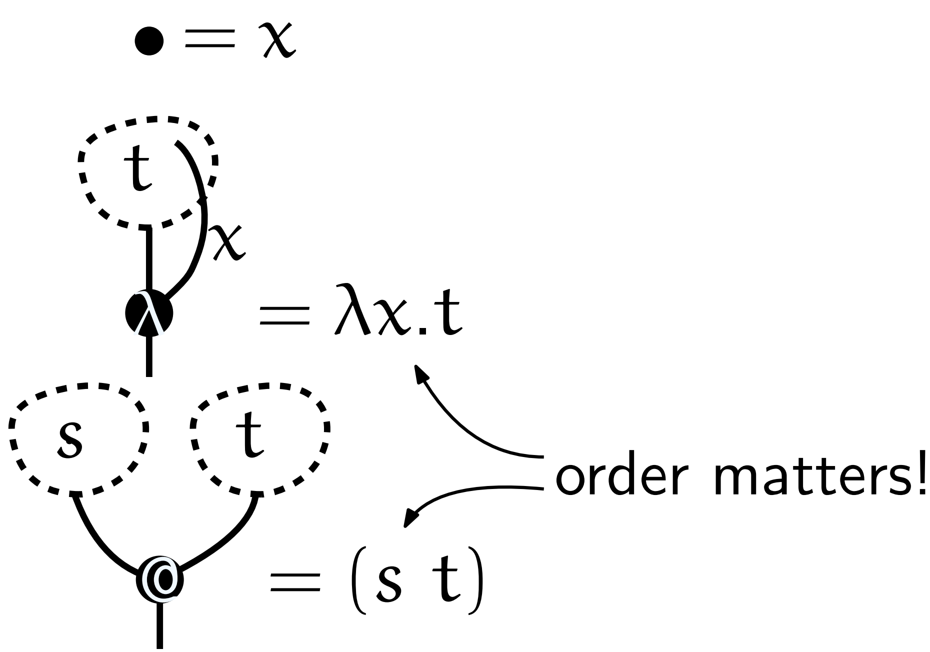

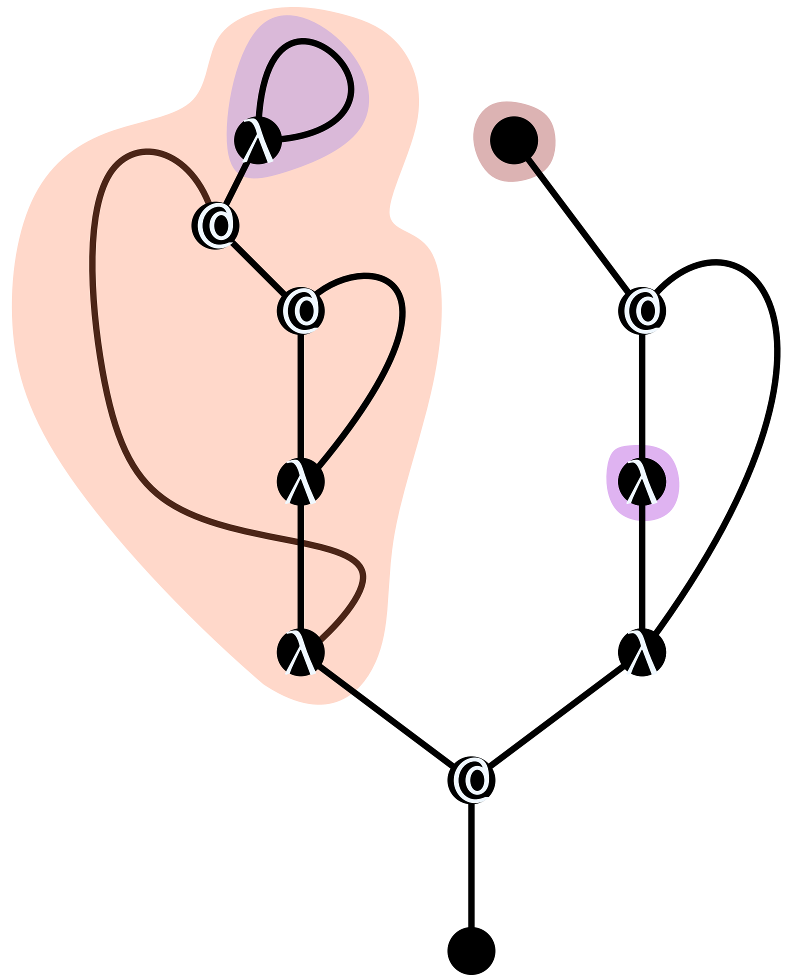

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Closed linear terms ↔ trivalent maps

Closed affine terms ↔ (2,3)-valent maps

Established in [BGJ13, BGGJ13]

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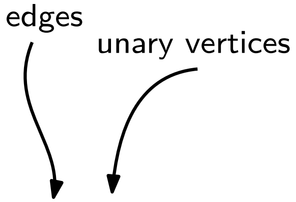

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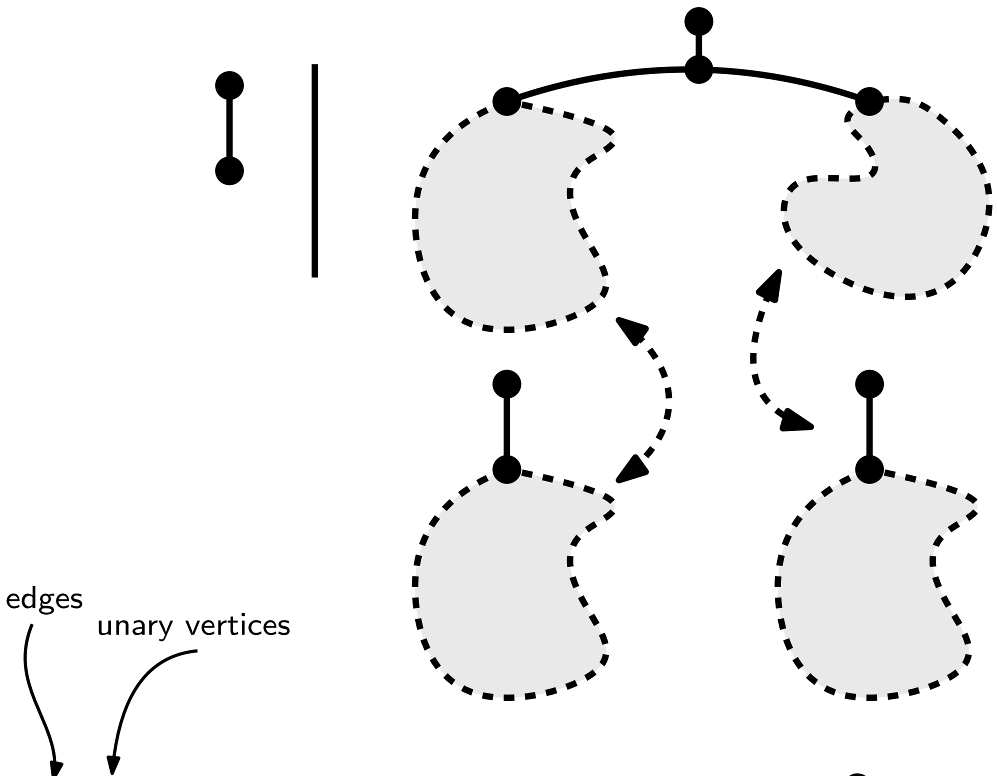

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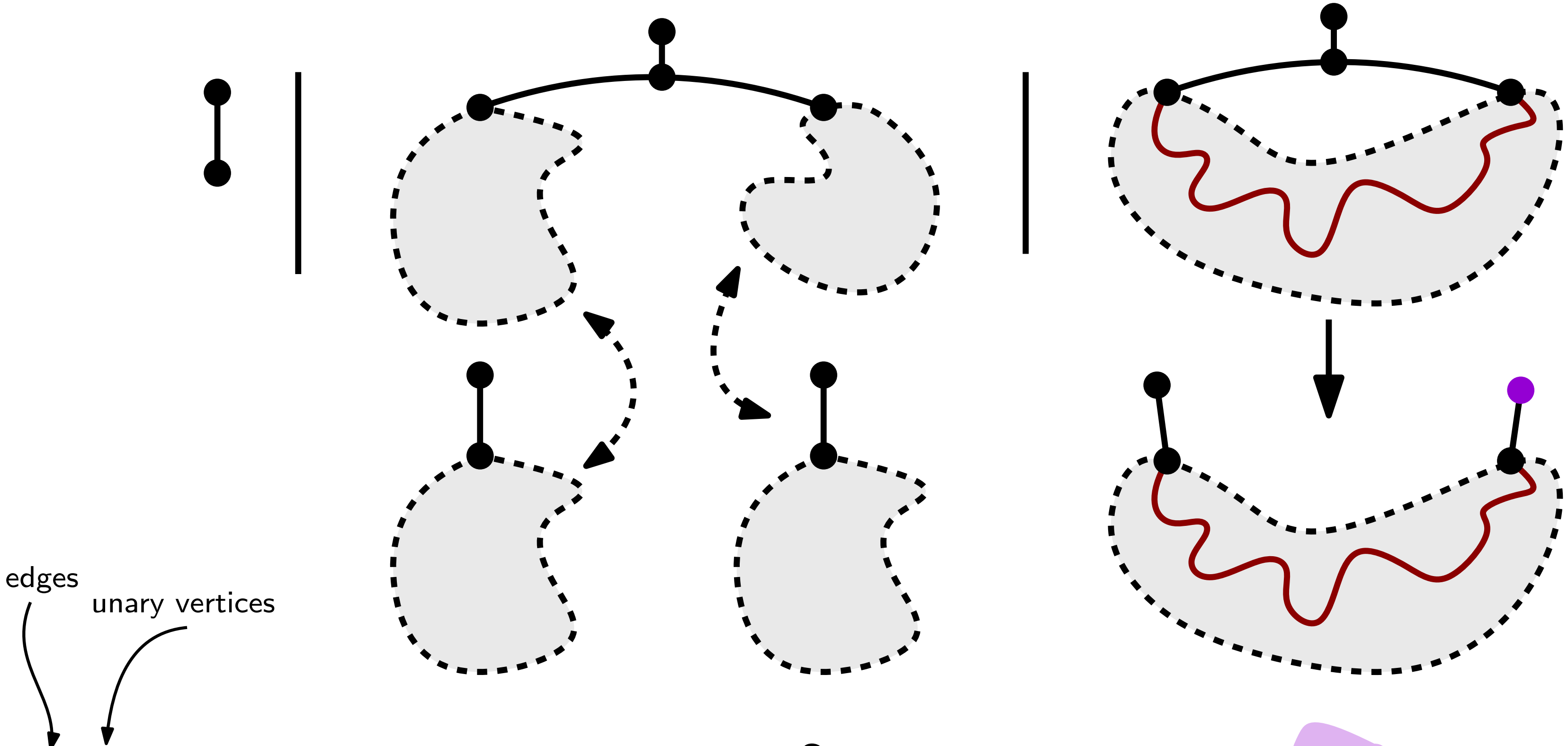

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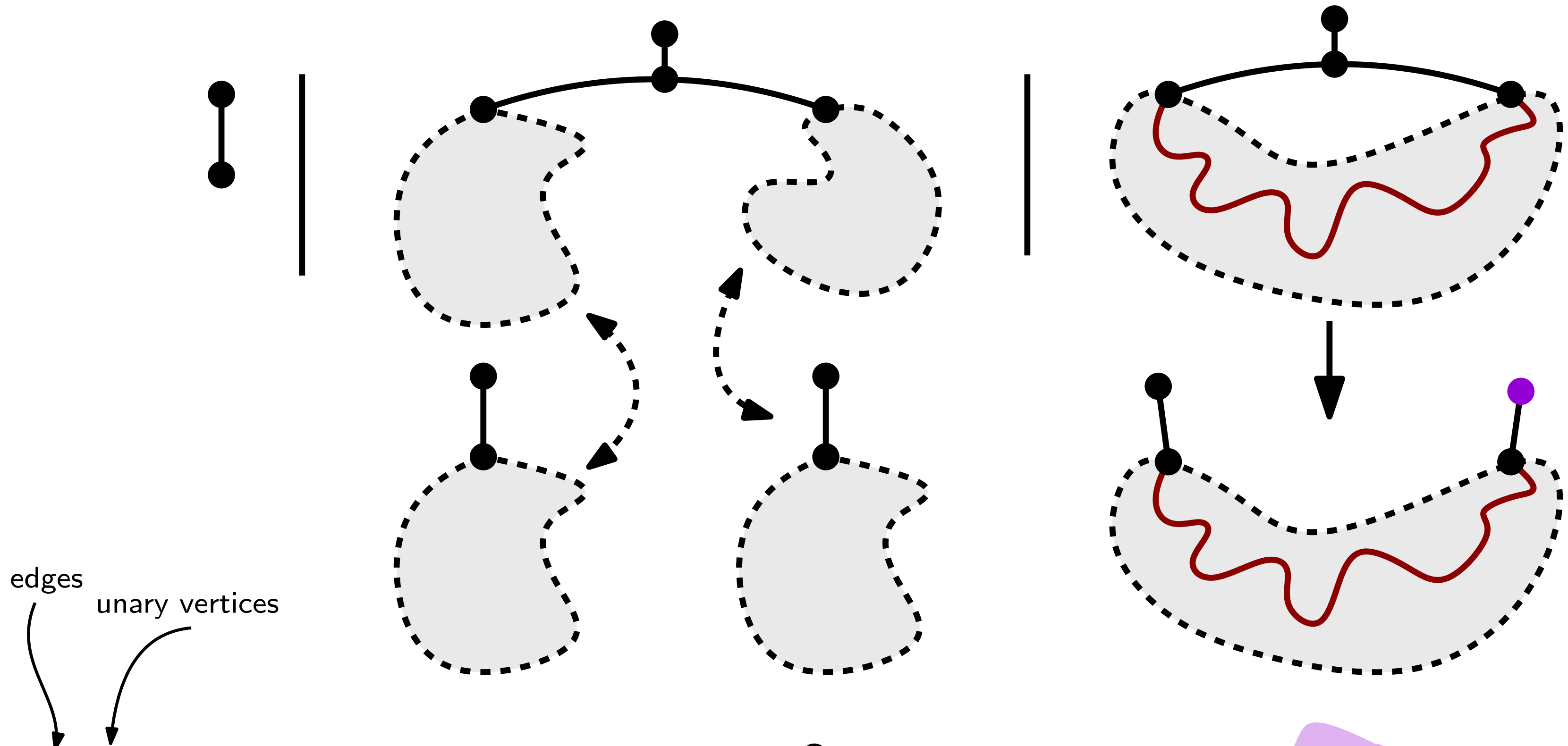

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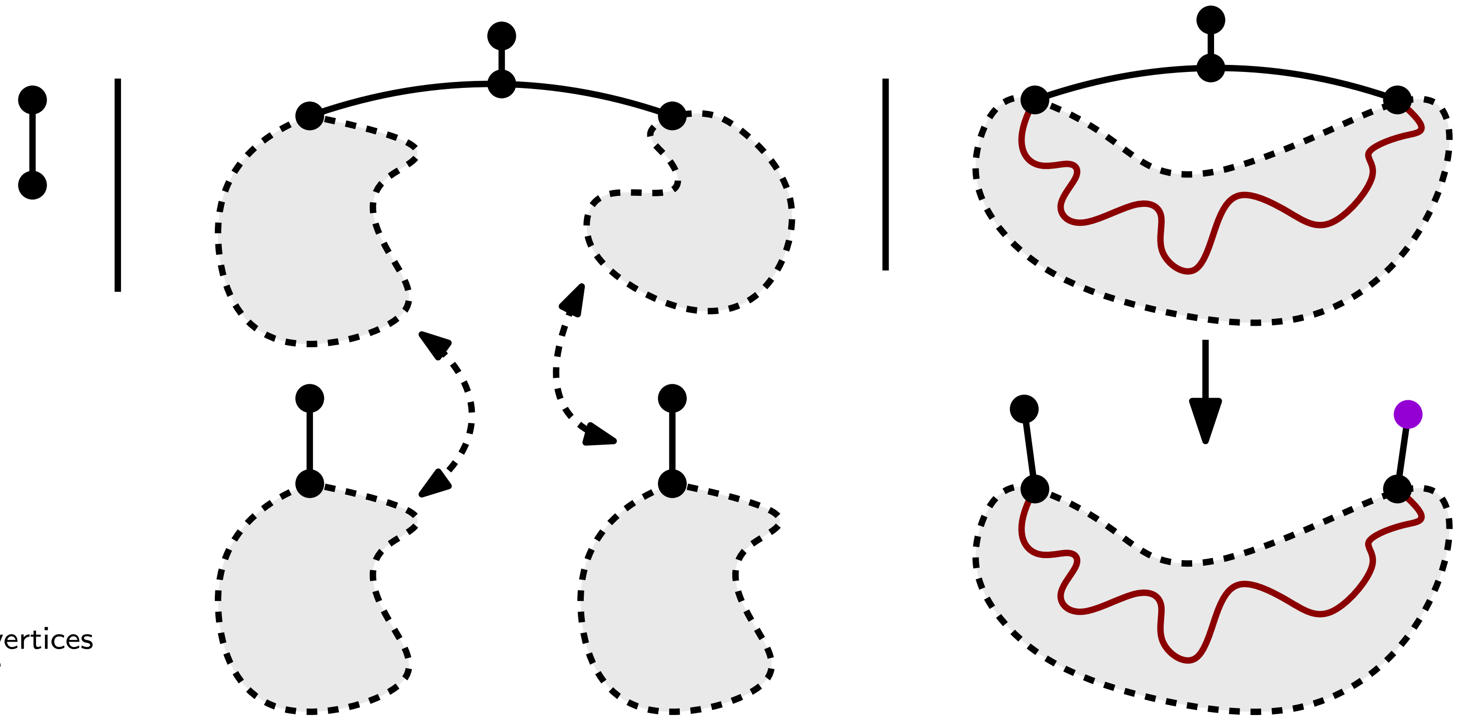

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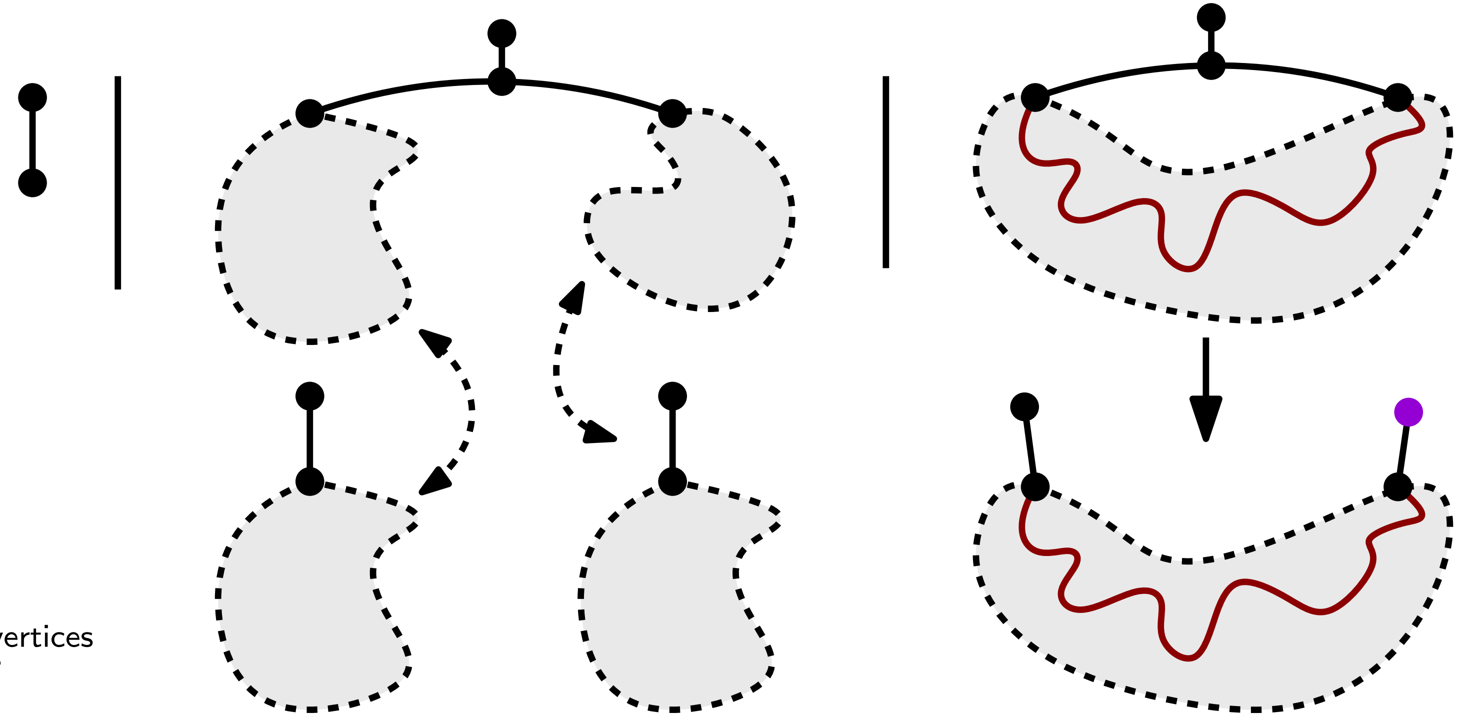

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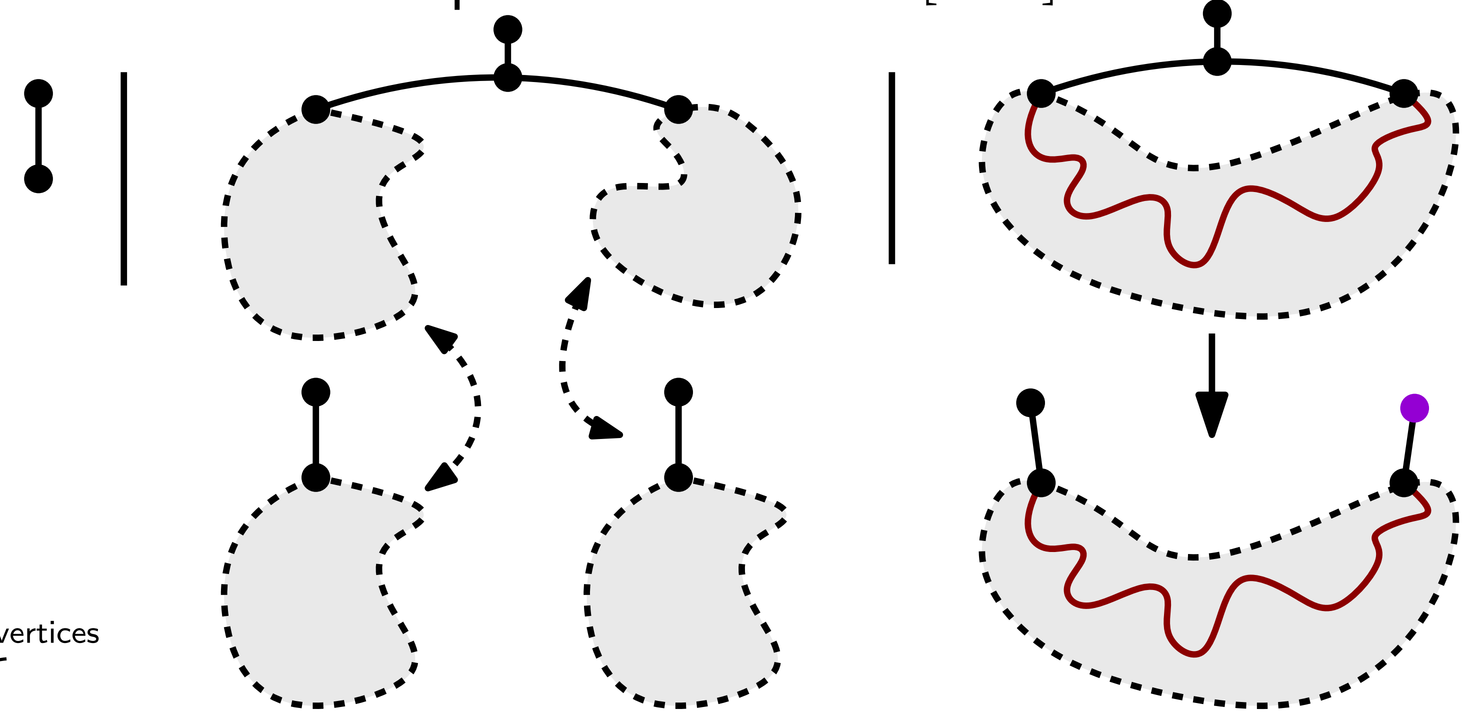
and open linear terms! [Z16]



edges, unary vertices

$$T(z, u) = z + zT(z)^2 + z\partial_u T(z)$$

subterms, free vars

$$\text{lin.term} = x \;|\; (s\ t) \;|\; \lambda x.t$$

Recap: λ-terms and maps

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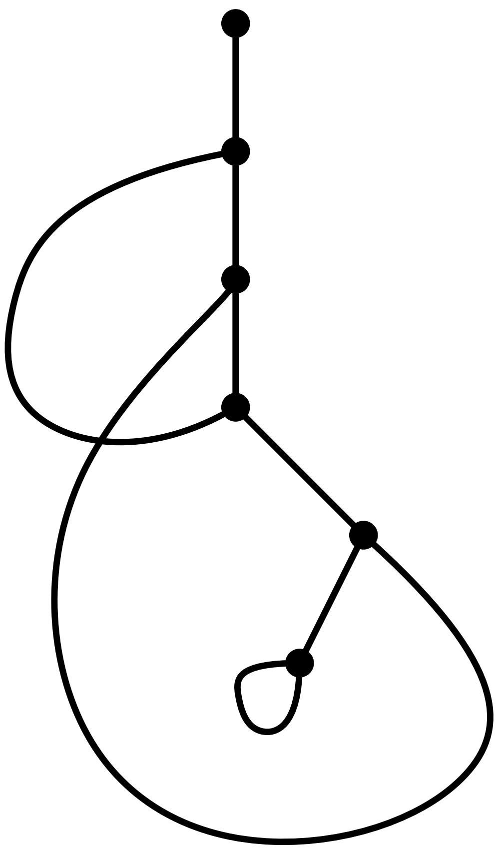
Our results: limit distributions

Closed trivalent maps $\leftrightarrow$ closed linear terms



$\lambda x.\lambda y.(y\ \lambda w.w)x$

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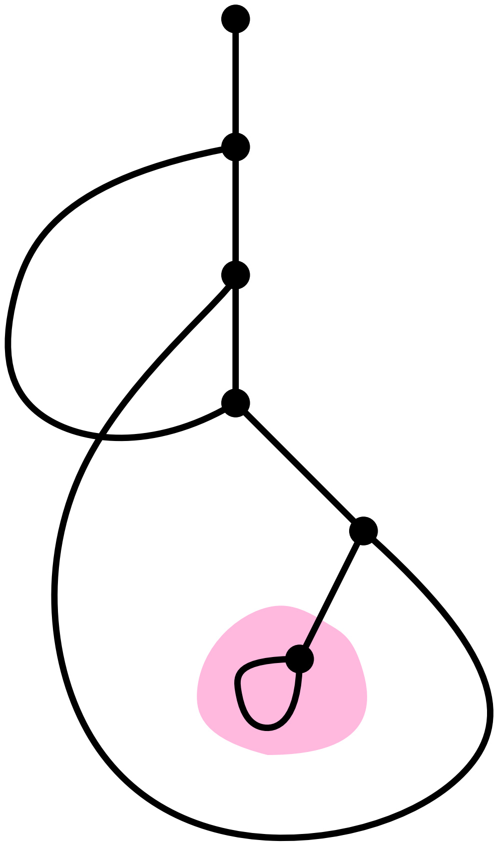

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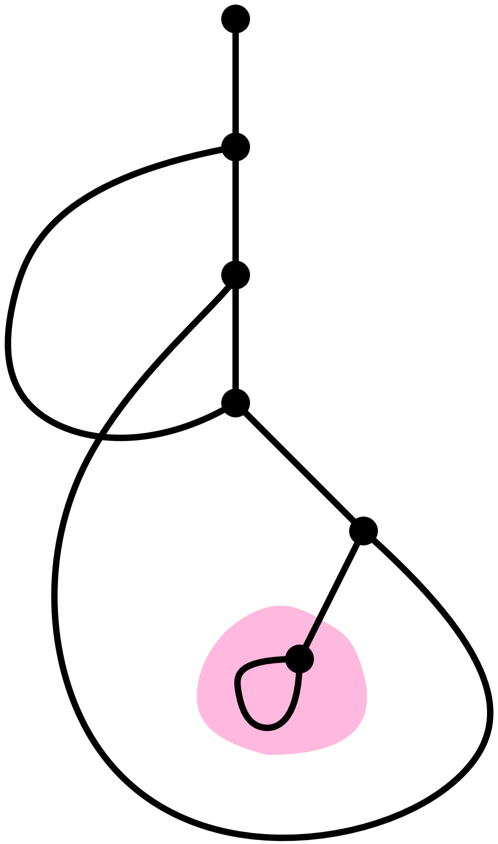

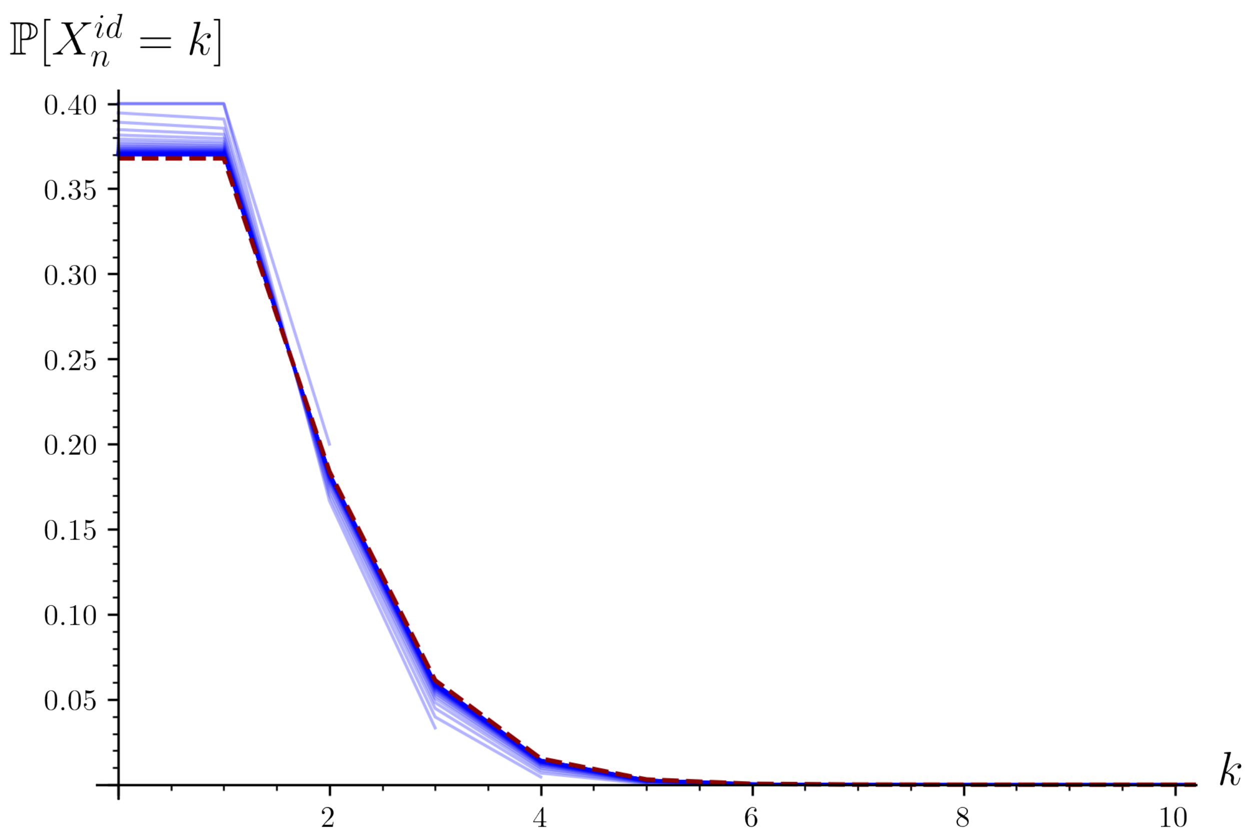

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$X_n^{id} \xrightarrow{D} \text{Poisson}(1)$

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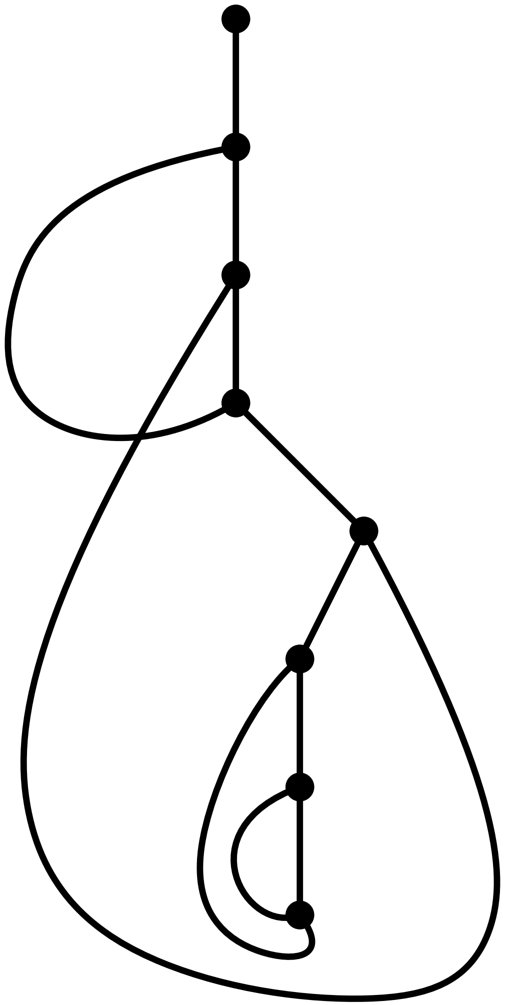

$$\lambda x.\lambda y.(y\ \lambda z.\lambda w.zw)x$$

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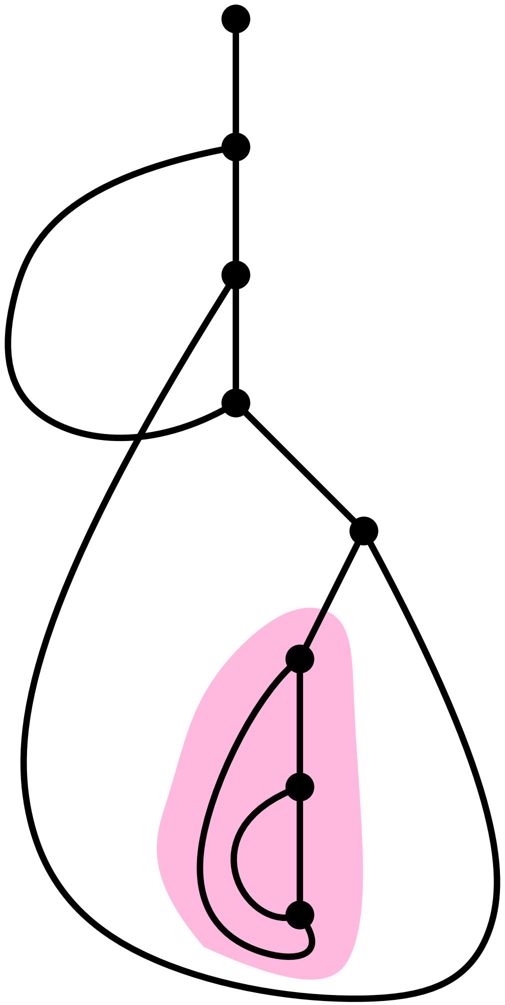

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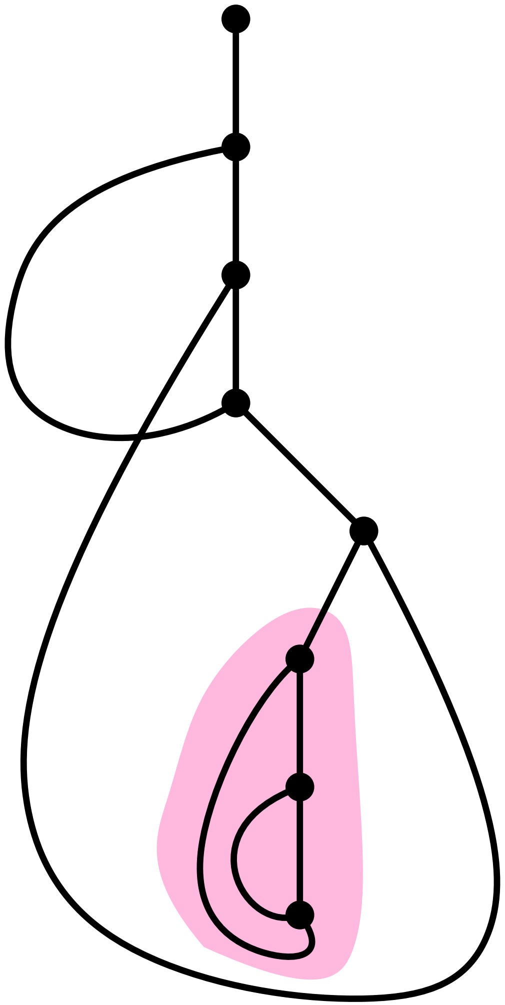
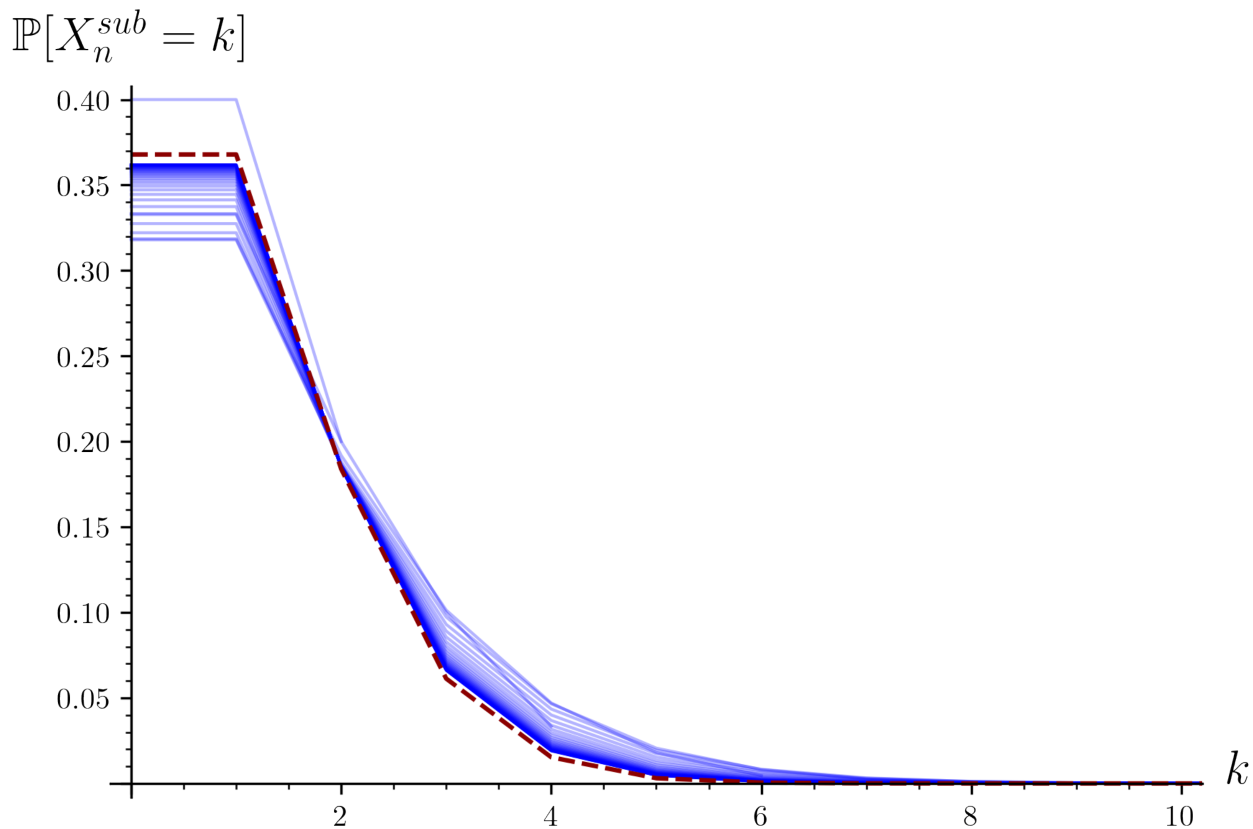


$\lambda x.\lambda y.(y\ \lambda z.\lambda w.zw)x$



$$X_n^{sub} \xrightarrow{D} \mathrm{Poisson}(1)$$

Our results: limit distributions

Closed trivalent maps $\leftrightarrow$ closed linear terms

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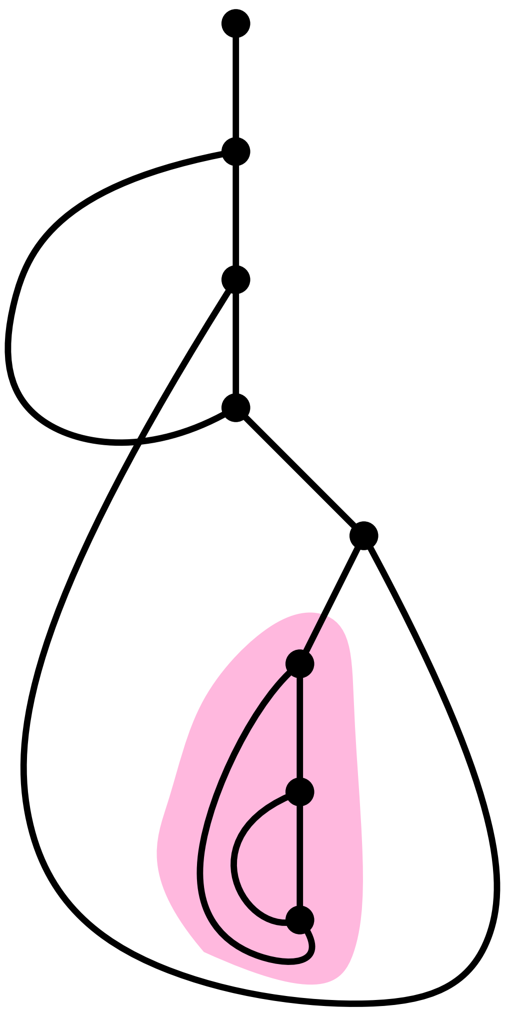
bad news for remote villages in rooted trivalent maps...

one bridge $\leftrightarrow$ no bridge



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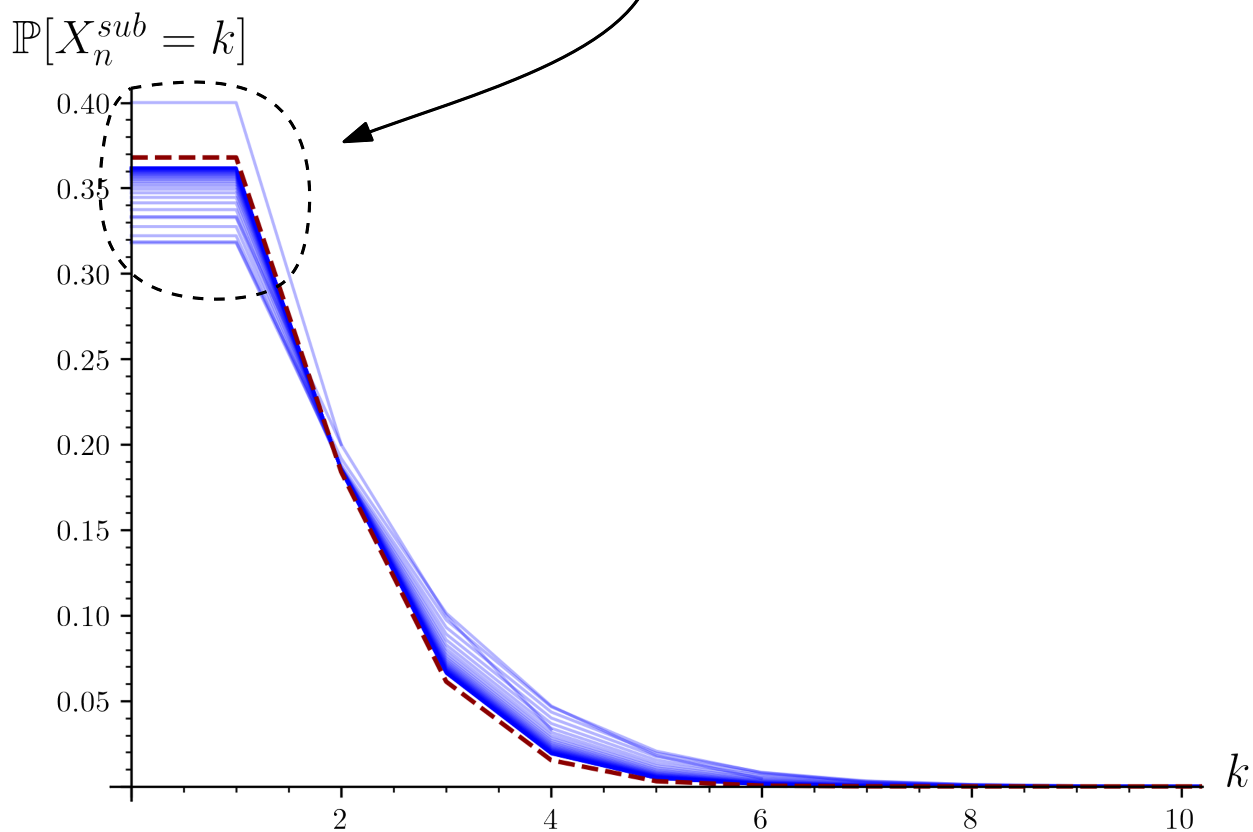

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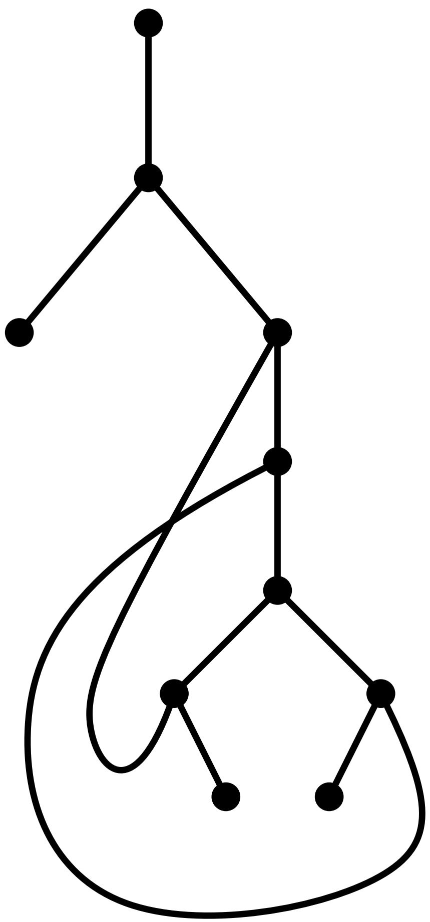
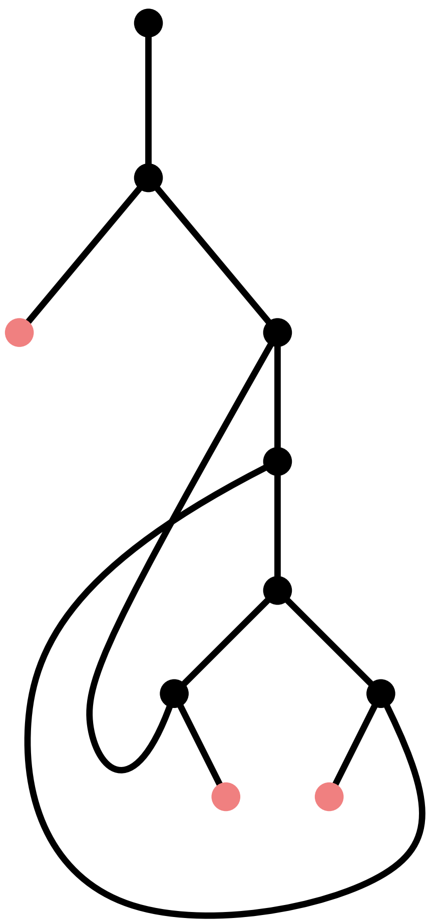


$$(a\ (\lambda x.\lambda y.(y\ b)(c\ x)))$$

Our results: limit distributions

Open trivalent maps ↔ open linear terms

\# unary vertices = # free vars



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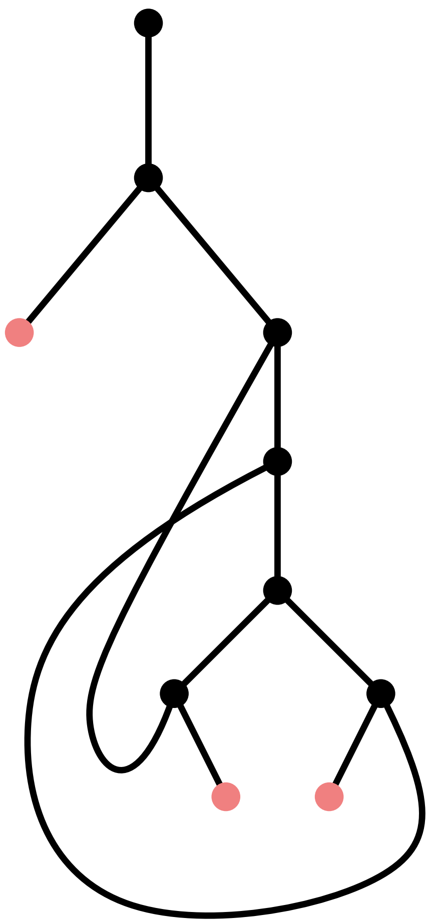

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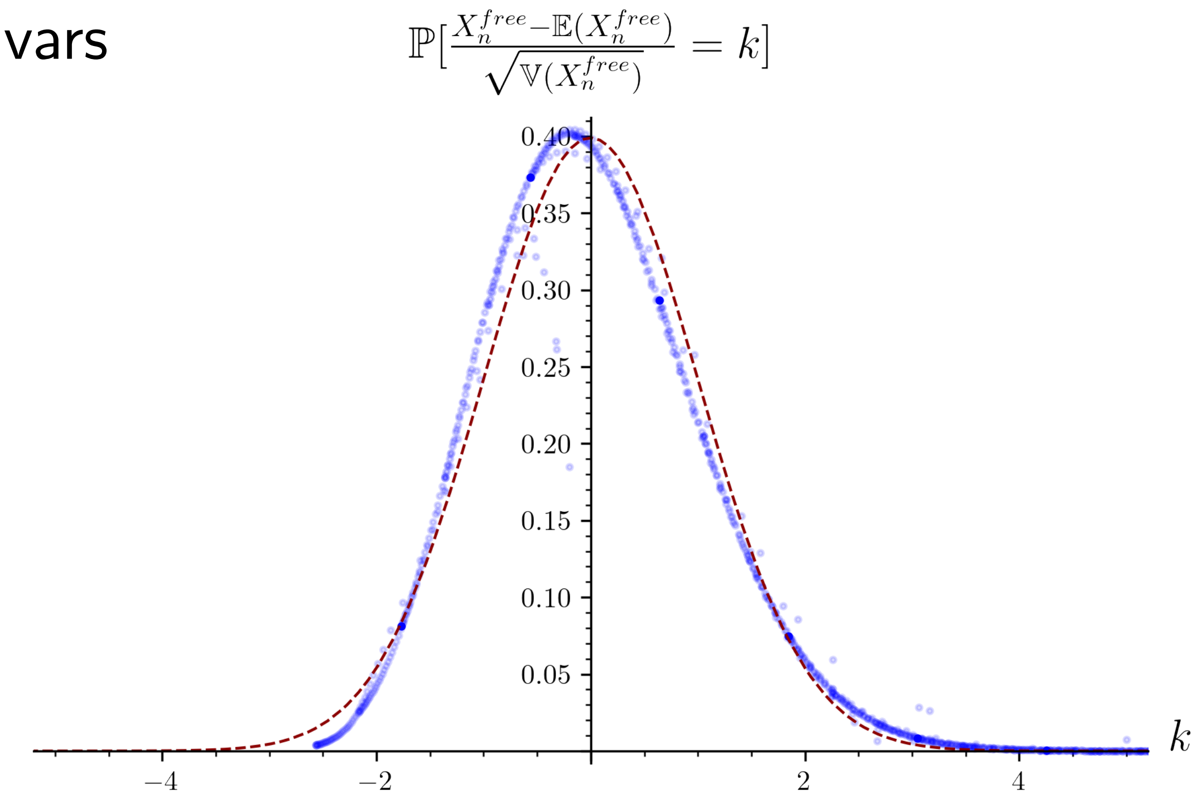
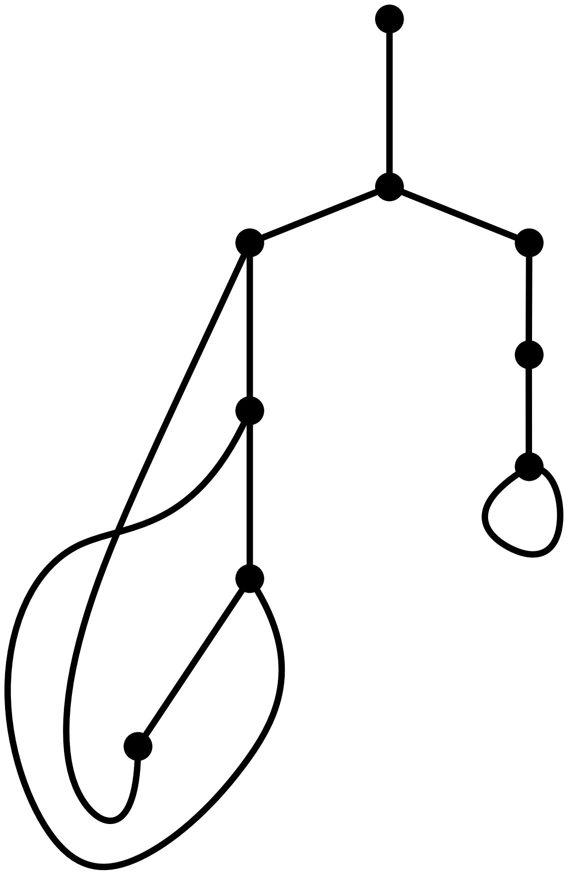


$$\frac{X_n^{free} - \mu_n}{\sqrt{\sigma_n^2}} \xrightarrow{D} \mathcal{N}(0,1)$$

for $\mu = \sigma^2 = (2n)^{1/3}$

Our results: limit distributions

(2,3)-valent maps $\leftrightarrow$ closed affine terms



$(\lambda x.\lambda y.(\lambda z.x)y)(\lambda w.\lambda v.\lambda u.u)$

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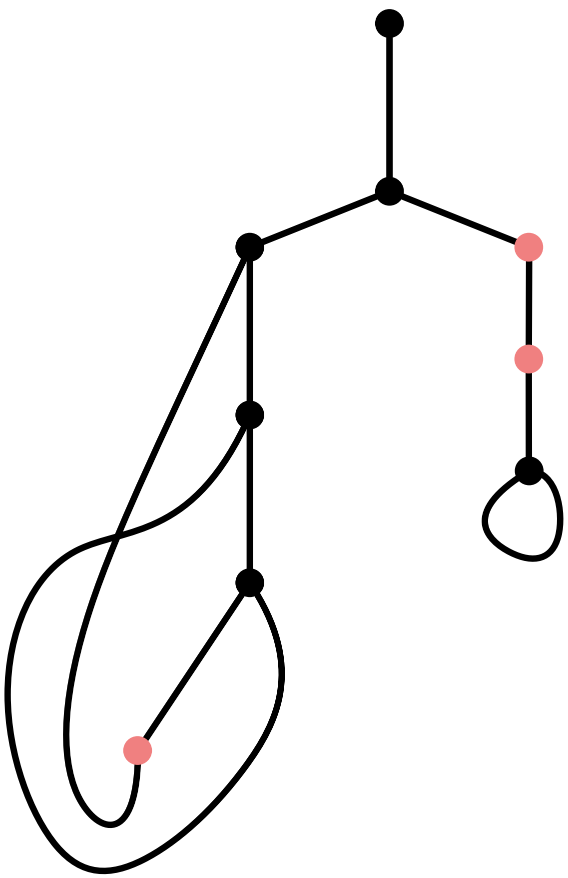

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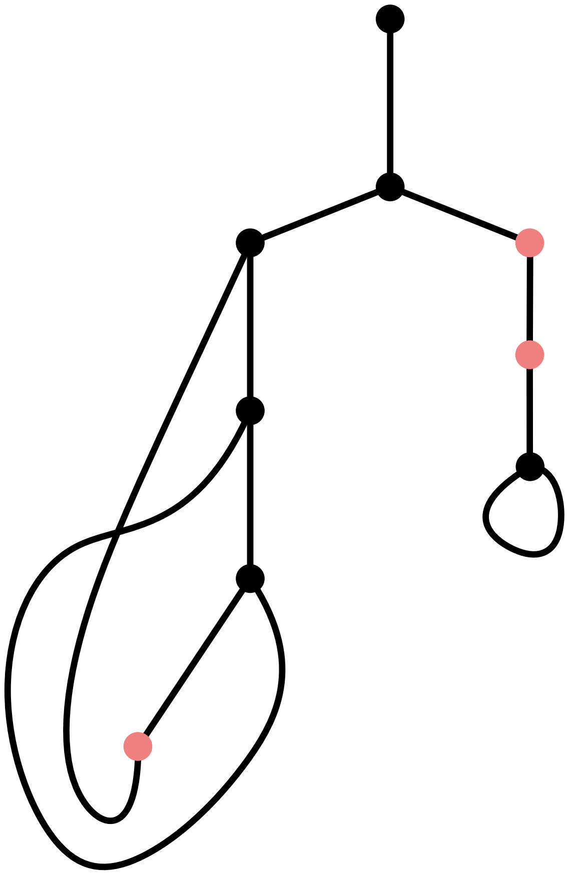

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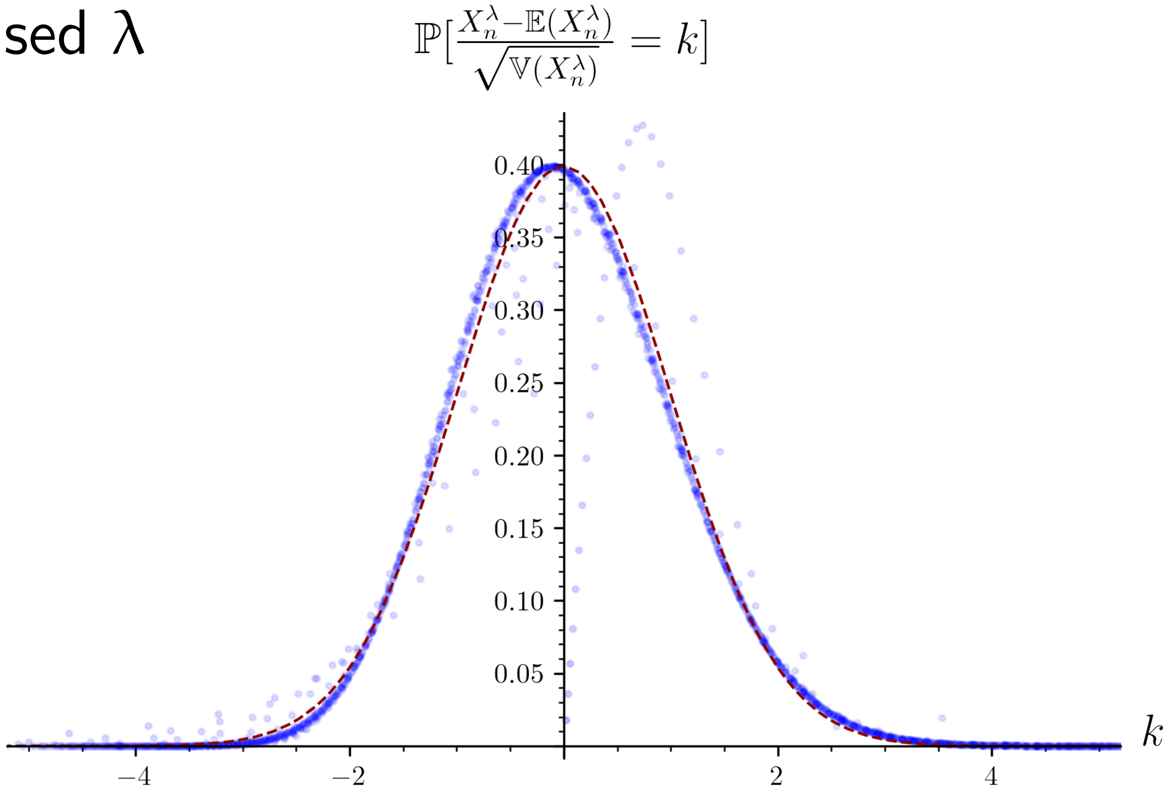

$$\frac{X_n^\lambda - \mu_n}{\sqrt{\sigma_n^2}} \xrightarrow{D} \mathcal{N}(0,1)$$

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- Schema based on compositions (see also [B75,FS93,B18,P19,BKW21]):

$F(z, u, G(z, u))$ inherits the limit law of $G(z, u)$

Proof sketch for loops/id-subterms:

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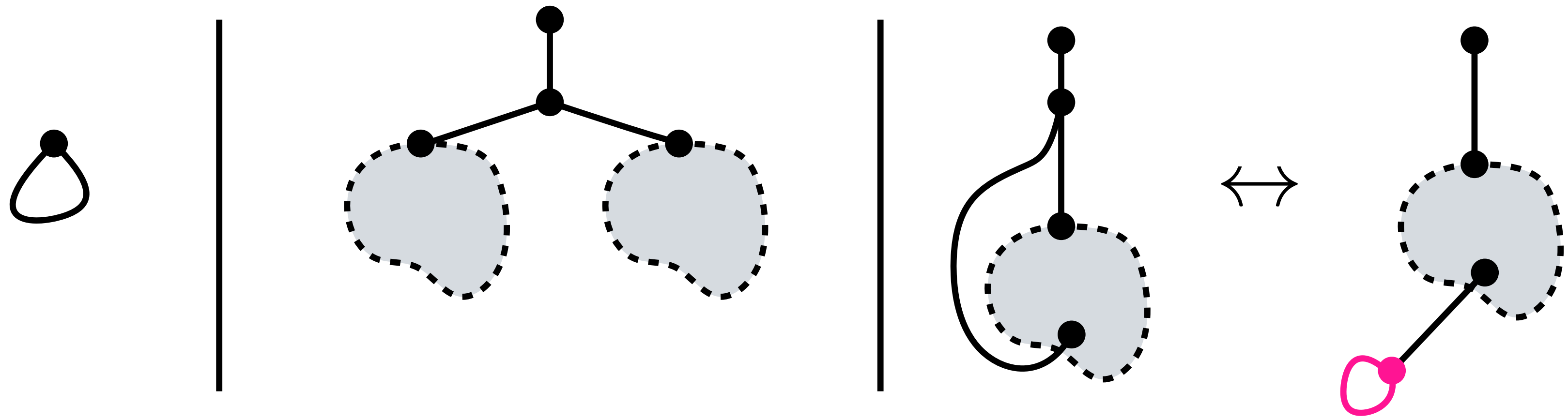
$$T_0^{\mathrm{id}}(z,u) = (u-1)z^2 + zT_0^{\mathrm{id}}(z,u)^2 + \partial_u T_0^{\mathrm{id}}(z,u)$$



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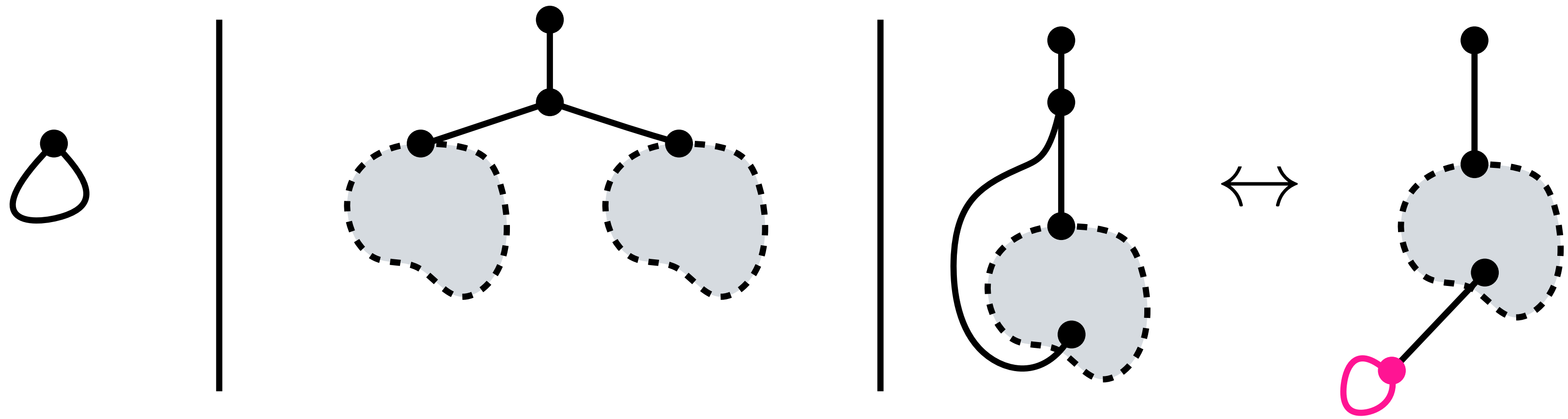

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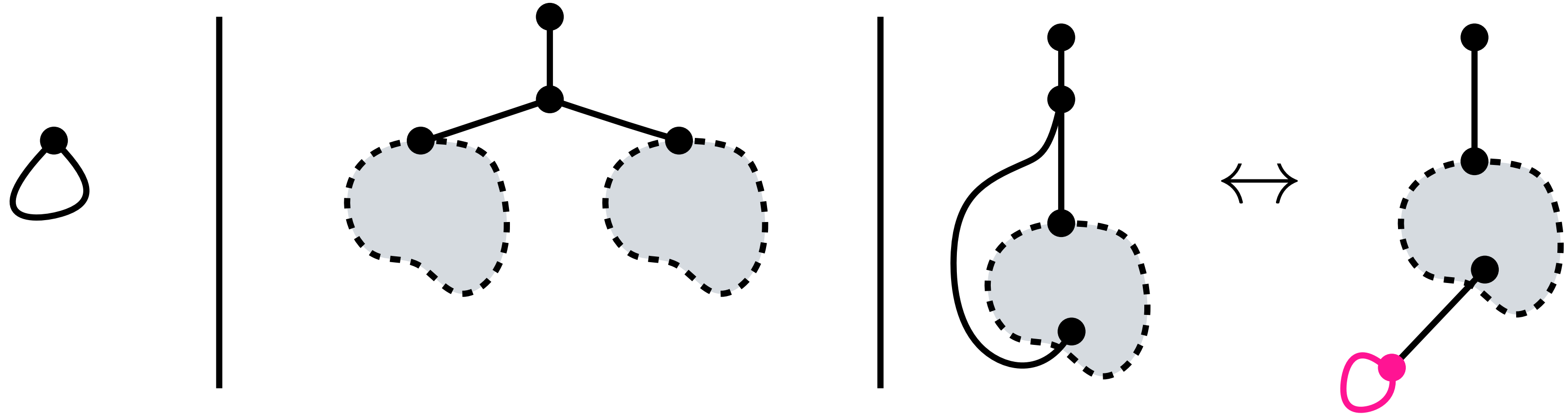

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$$[z^n] \quad \partial_u T_0^{\mathrm{id}}\big|_{v=1} = T_0^{\mathrm{id}} - (u-1)z^2 - z(T_0^{\mathrm{id}})^2 \qquad \sim [z^n]T_0^{\mathrm{id}}(z,1)$$

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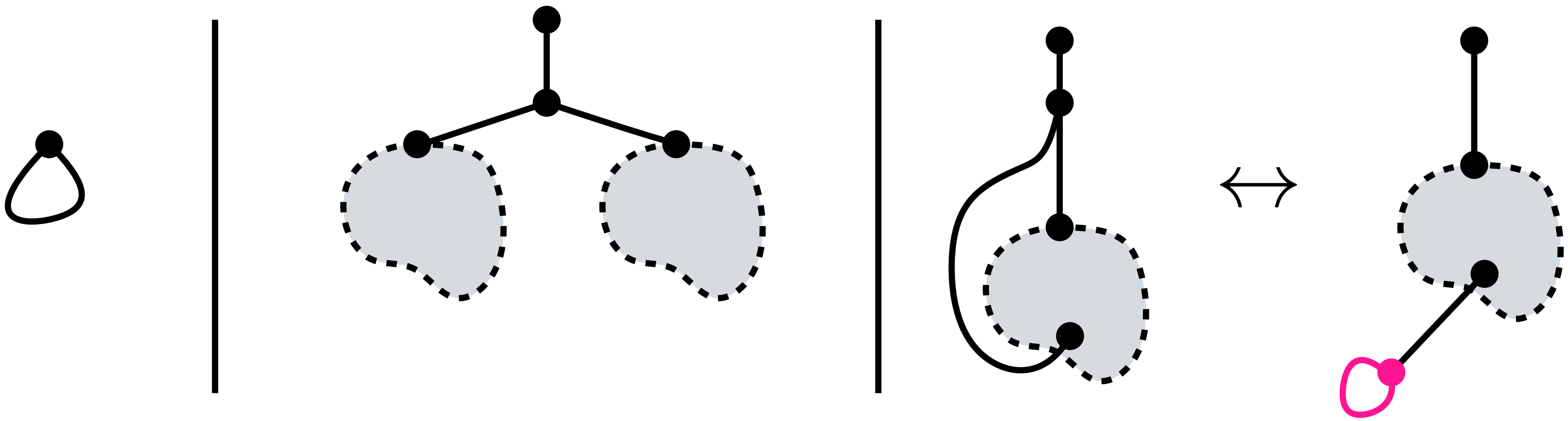

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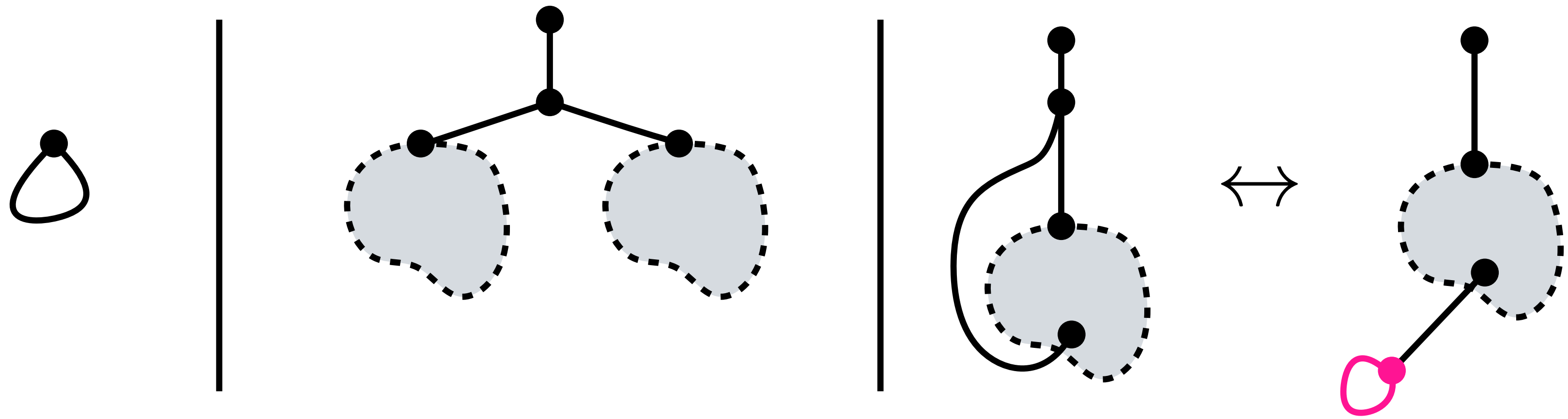

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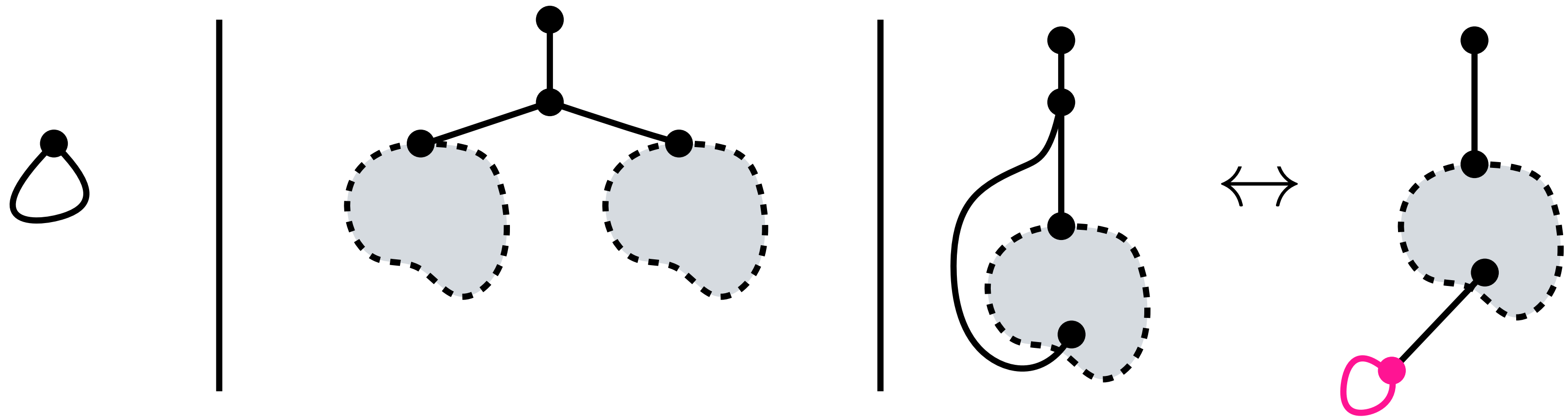

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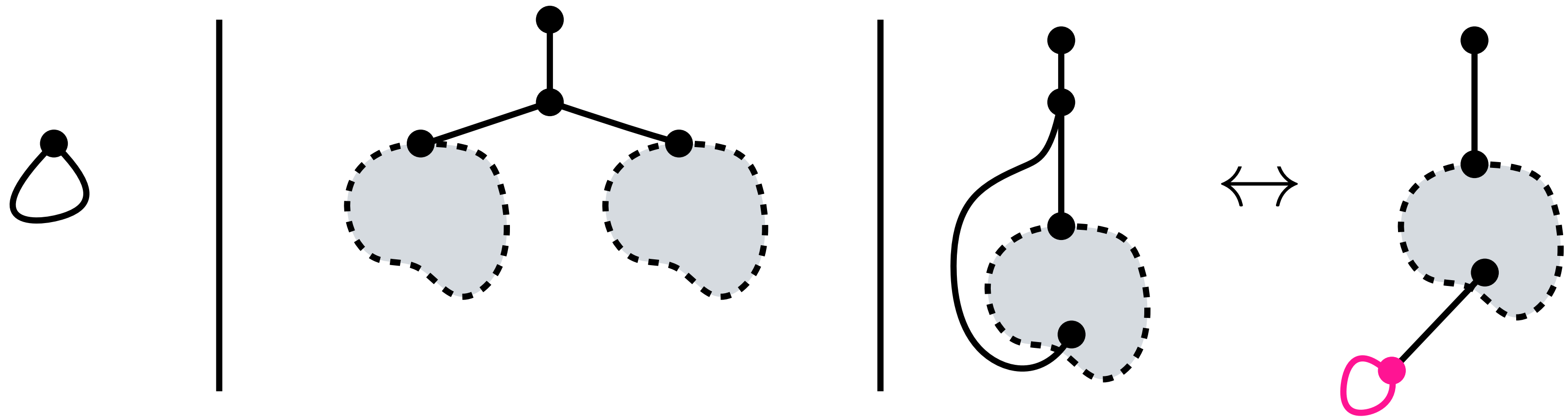

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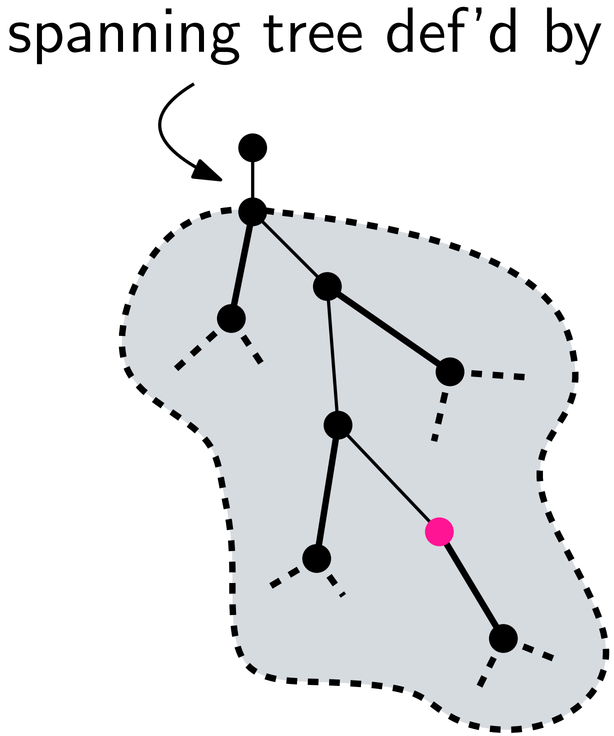
Schema then yields Poisson(1) limit law

Proof sketch for bridges/closed subterms:



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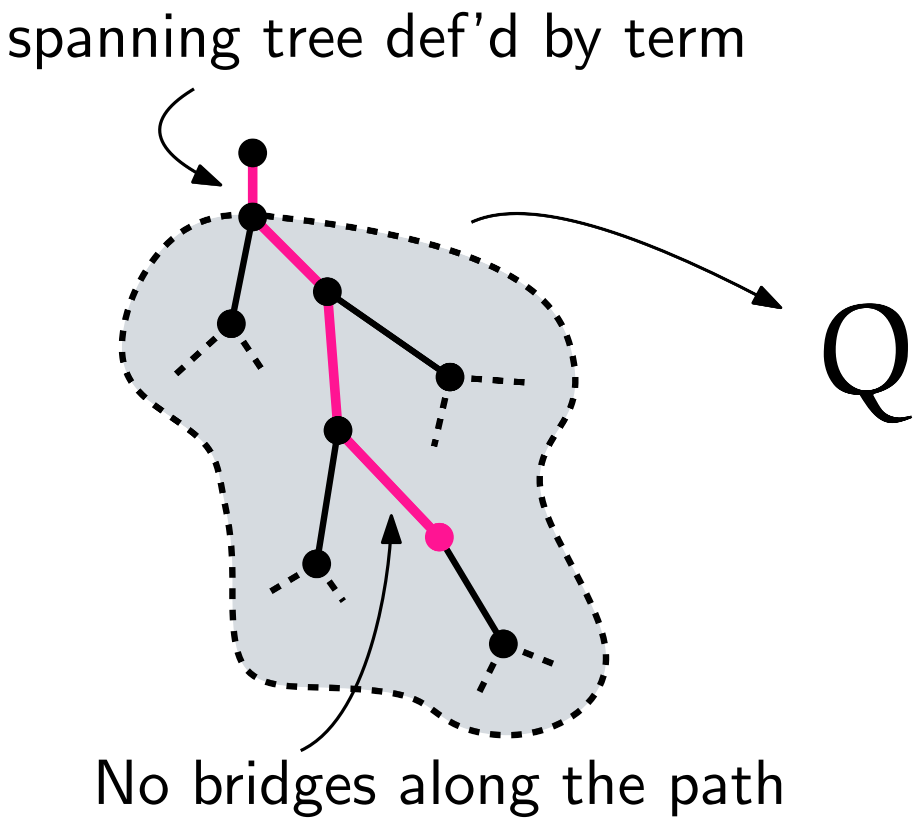

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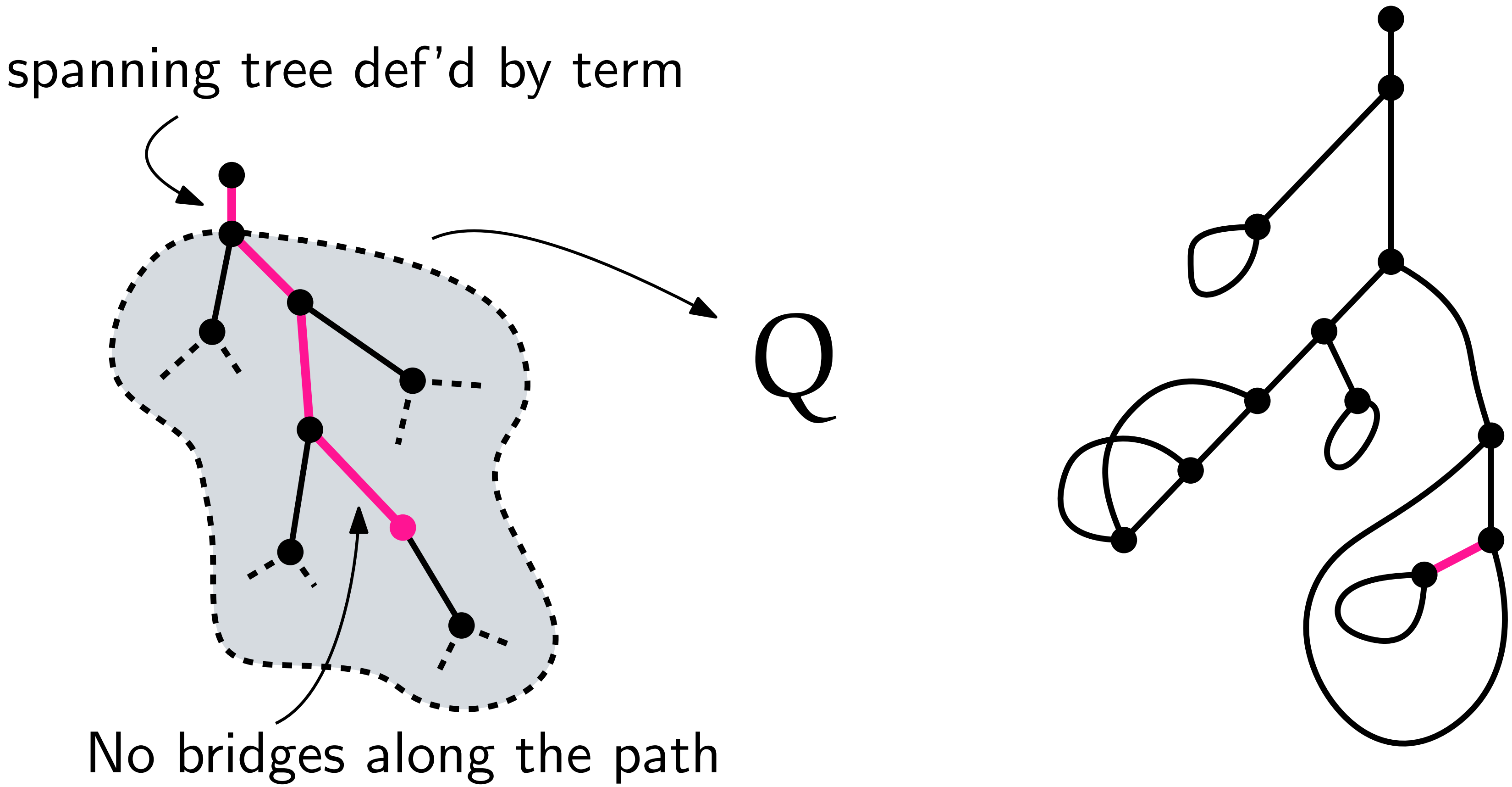

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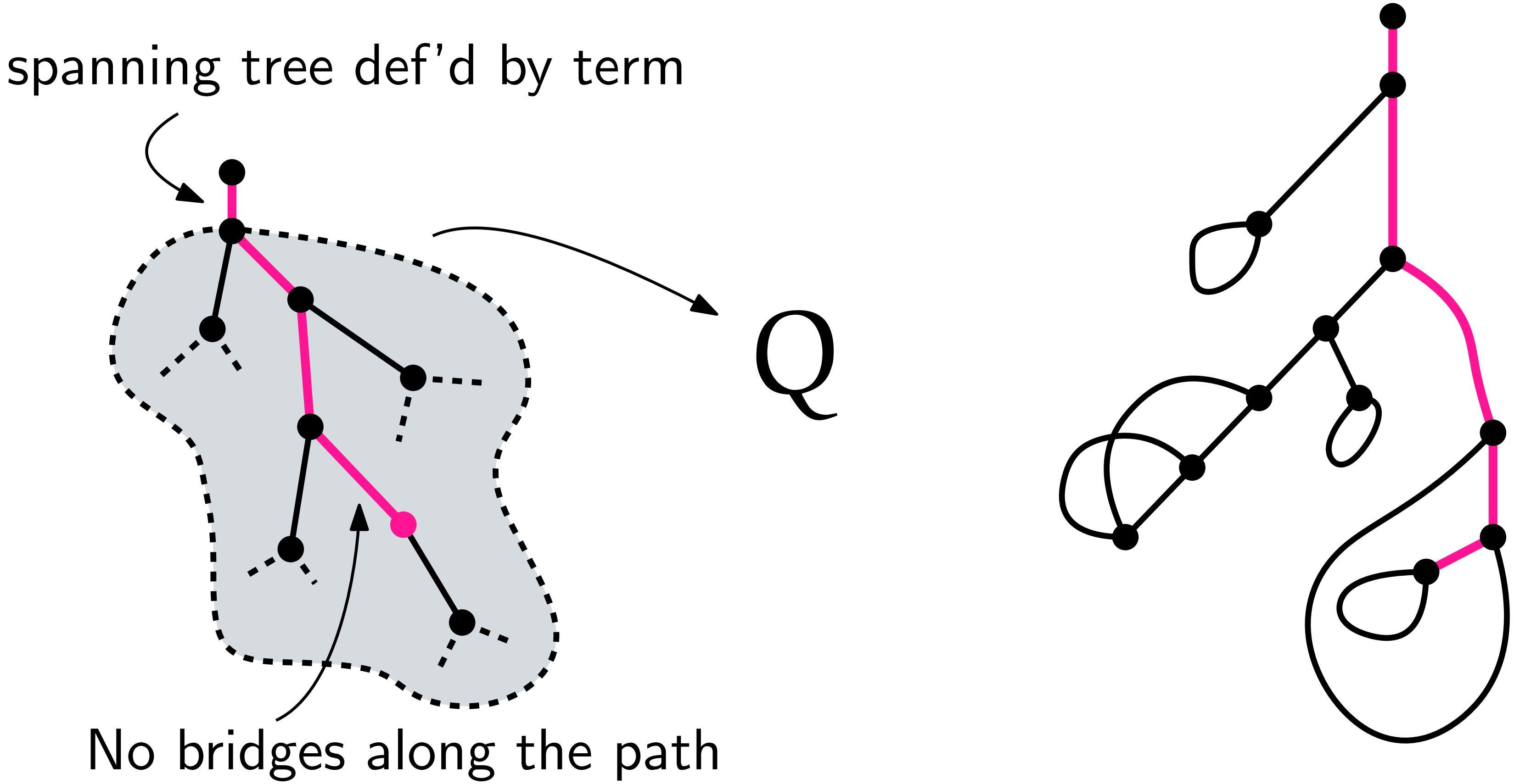

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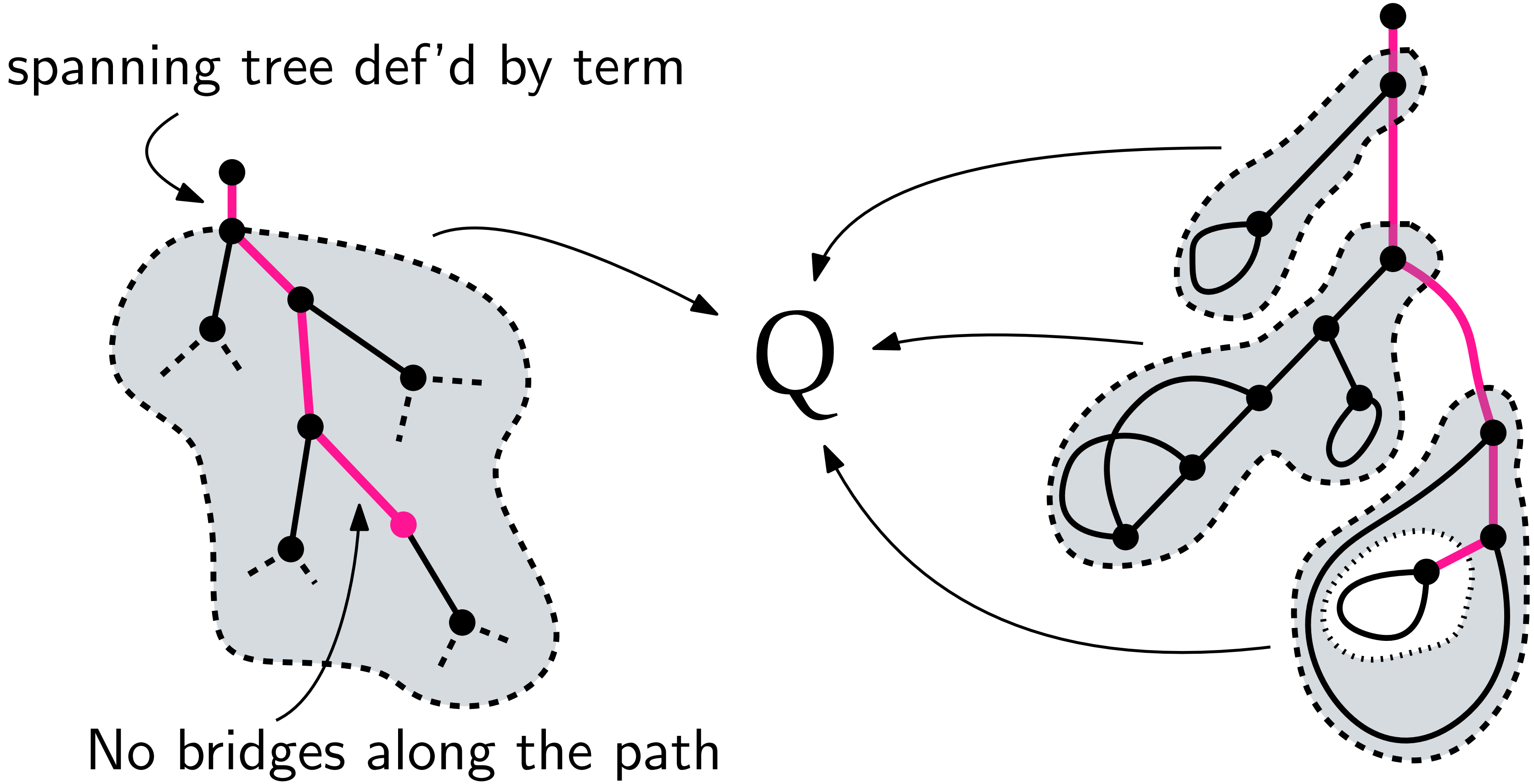

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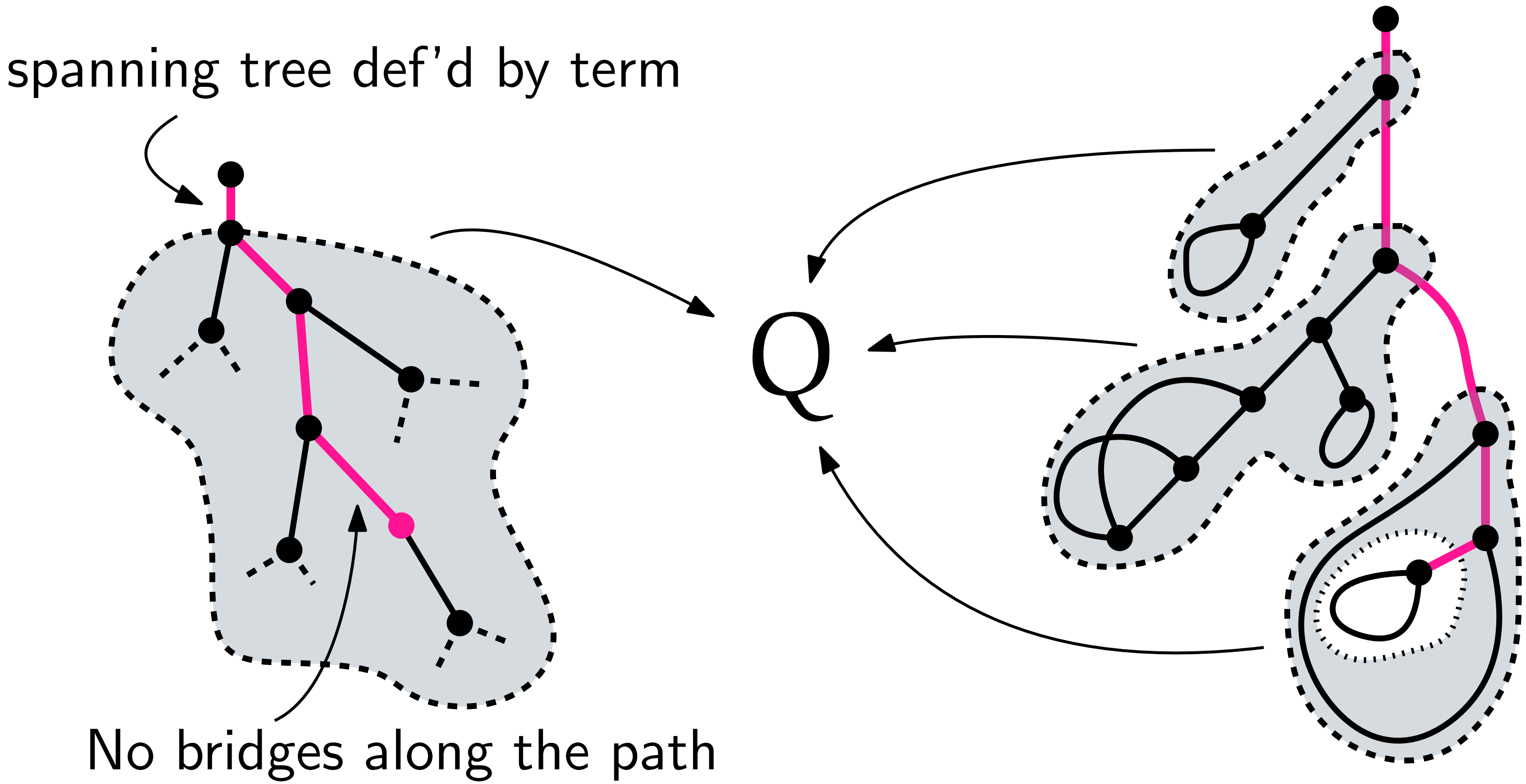

$$\frac{\partial}{\partial v} T_0^{sub}(z,v) = -\frac{v^2 z T_0^{sub}(z,v)^3 + z^2 T_0^{sub}(z,v) - T_0^{sub}(z,v)^2}{(v^3 - v^2) z T_0^{sub}(z,v)^2 + v z^2 - (v-1) T_0^{sub}(z,v)}$$

May be pumped using our schema

Proof sketch for vertices of given degree:

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Specifications based on exponential Hadamard products

$$\mathrm{OT}(z, u) = uz^2 + z^4 + z^5 \frac{\partial}{\partial z} \left(\ln \left(\exp(z^2/2) \odot \exp(z^3/3 + uz)\right)\right)$$

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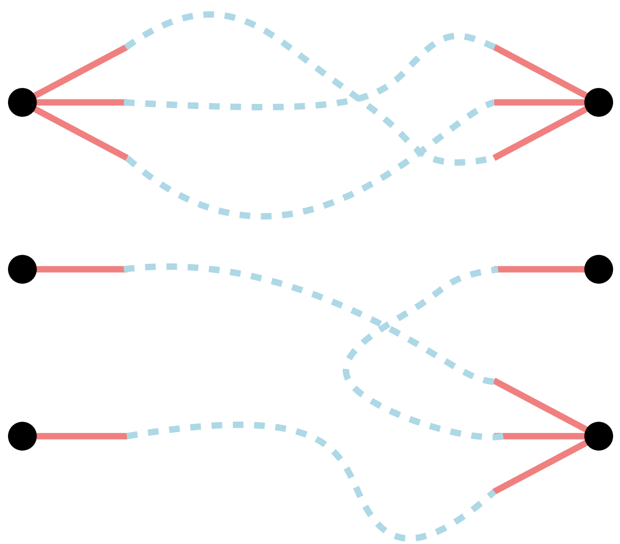

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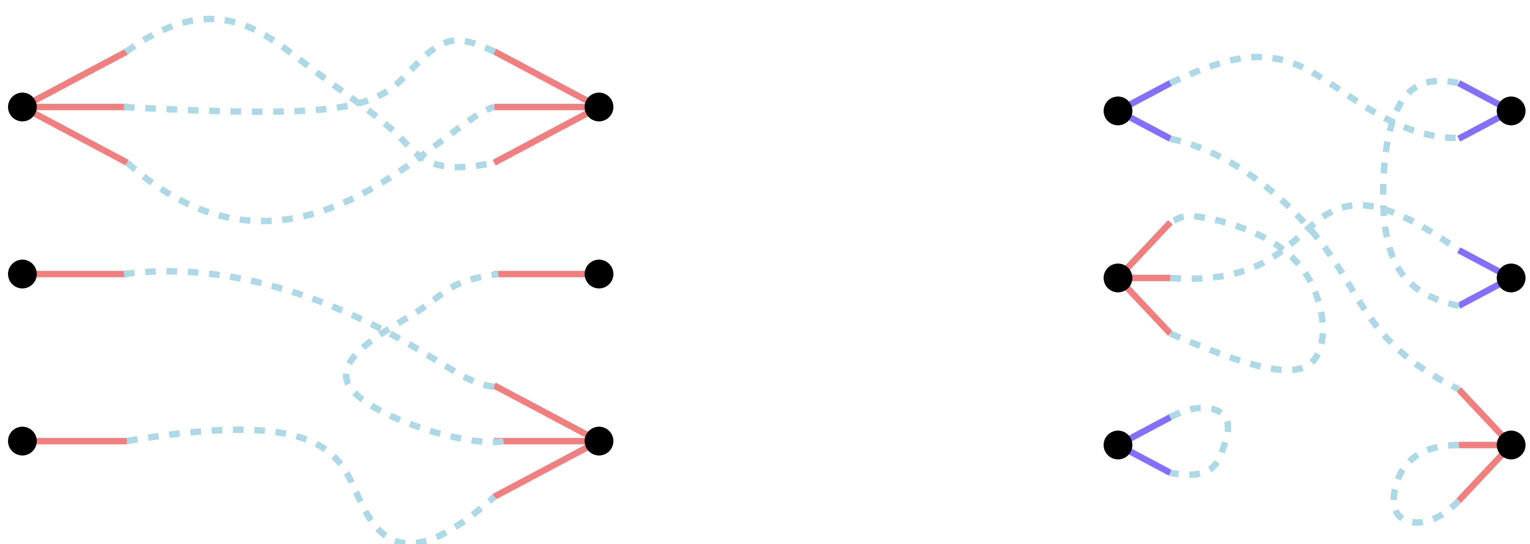

(2,3)-valent maps

$$\mathrm{TT}(z,u) = z\frac{\partial}{\partial z}\left(\ln\left(\exp\left(\frac{z^2}{2}\right) \odot \exp\left(\frac{z^3}{3} + \frac{uz^2}{2}\right)\right)\right)$$

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closed affine terms

Compositions for fast-growing series:

$$F(z, u, G(z, u))$$

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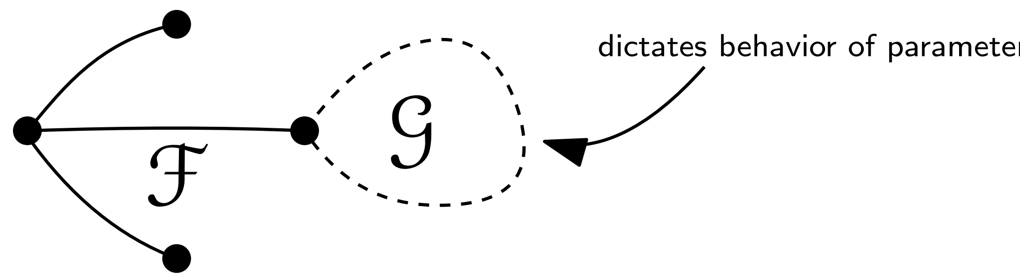

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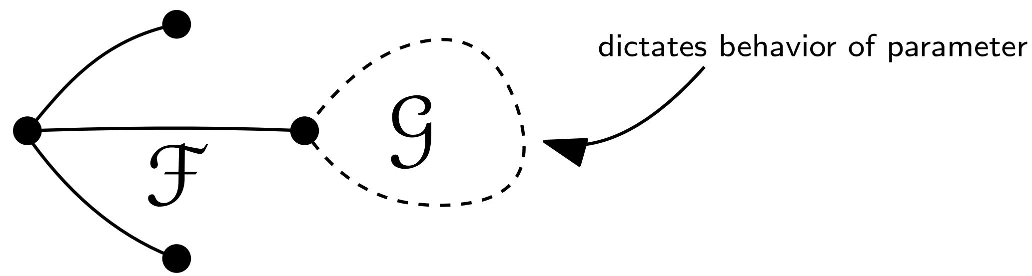

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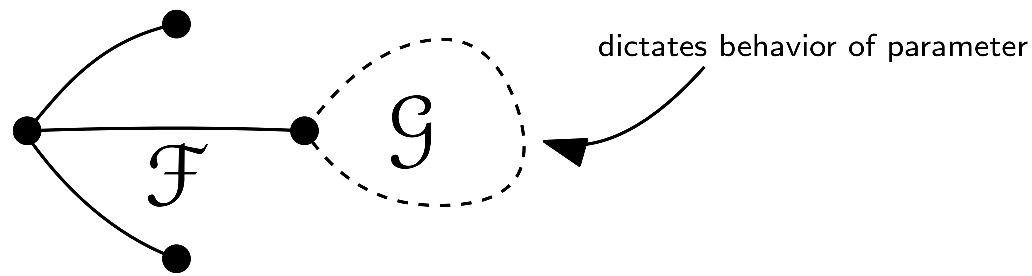

If F is the logarithm:

Asymptotically, almost all not-necessarily-connected $\mathcal{G}$-structures are connected, so the distribution of params. is the same for connected and not-necessarily-so structures!

Proof sketch for bridges/closed subterms (contd.) :

$$OT(z,u) = uz^2 + z^4 + z^5 \frac{\partial}{\partial z}\left(\ln\left(\exp(z^2/2) \odot \exp(z^3/3 + uz)\right)\right)$$

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Ammenable to saddle-point analysis!

Both yield Gaussian limit laws

Proof sketch for bridges/closed subterms (contd.) :

rooted → $\frac{\partial}{\partial z}$; connected → $\ln$

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Ammenable to saddle-point analysis!

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Use schema for compositions to show that the results carry over!

Mean number of β-redices in closed terms (WIP)

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- A standard decomposition for closed terms

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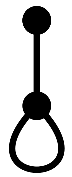
identity



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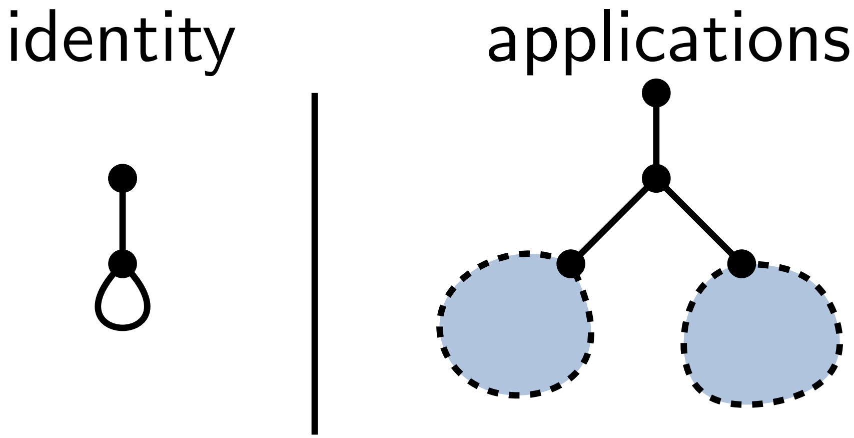

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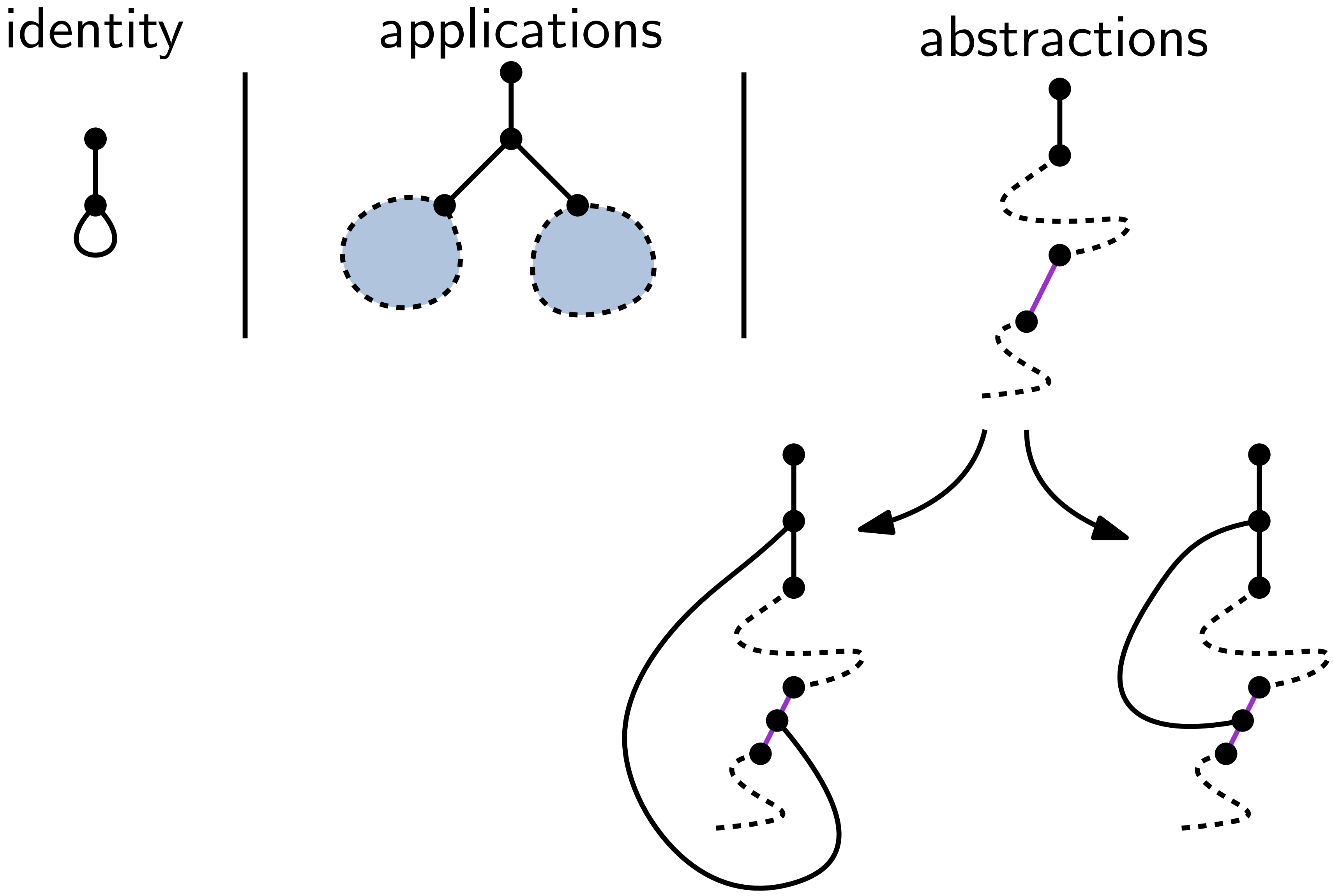

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- Tracking redices during the decomposition

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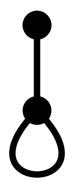
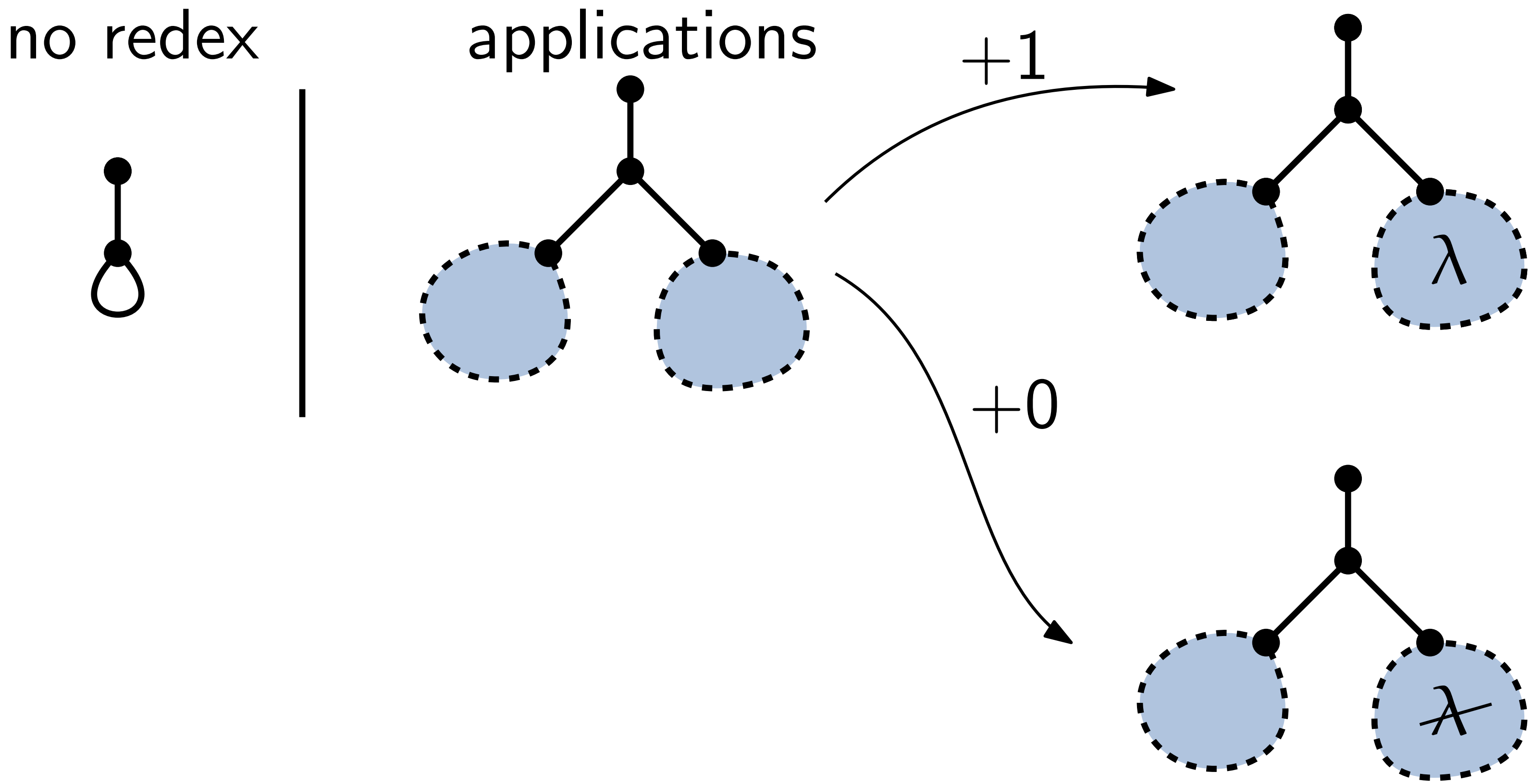
- Tracking redices during the decomposition

no redex



Mean number of β-redices in closed terms (WIP)

- Tracking redices during the decomposition

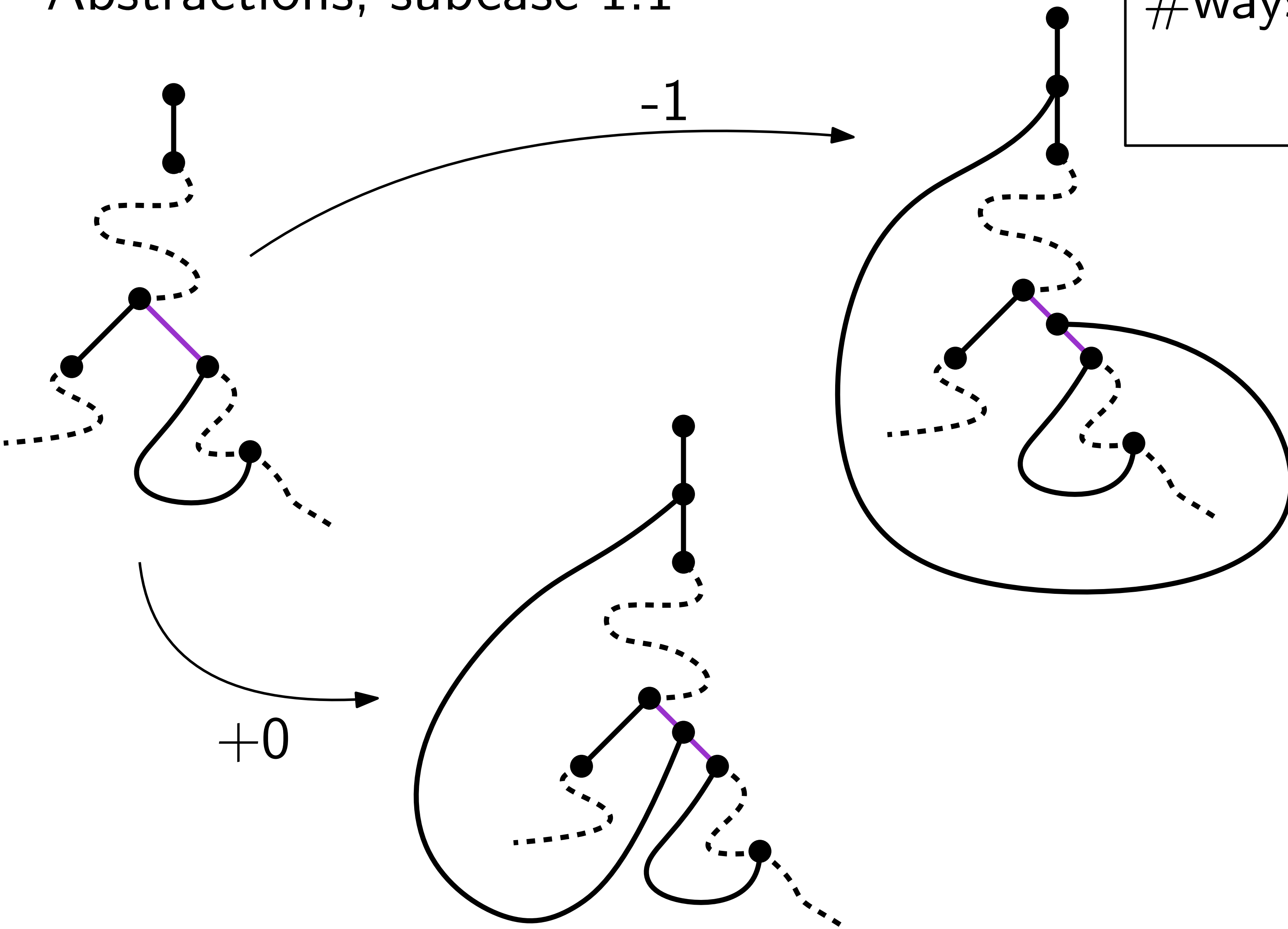


Mean number of β-redices in closed terms (WIP)

- Tracking redices during the decomposition

Abstractions, subcase 1.1

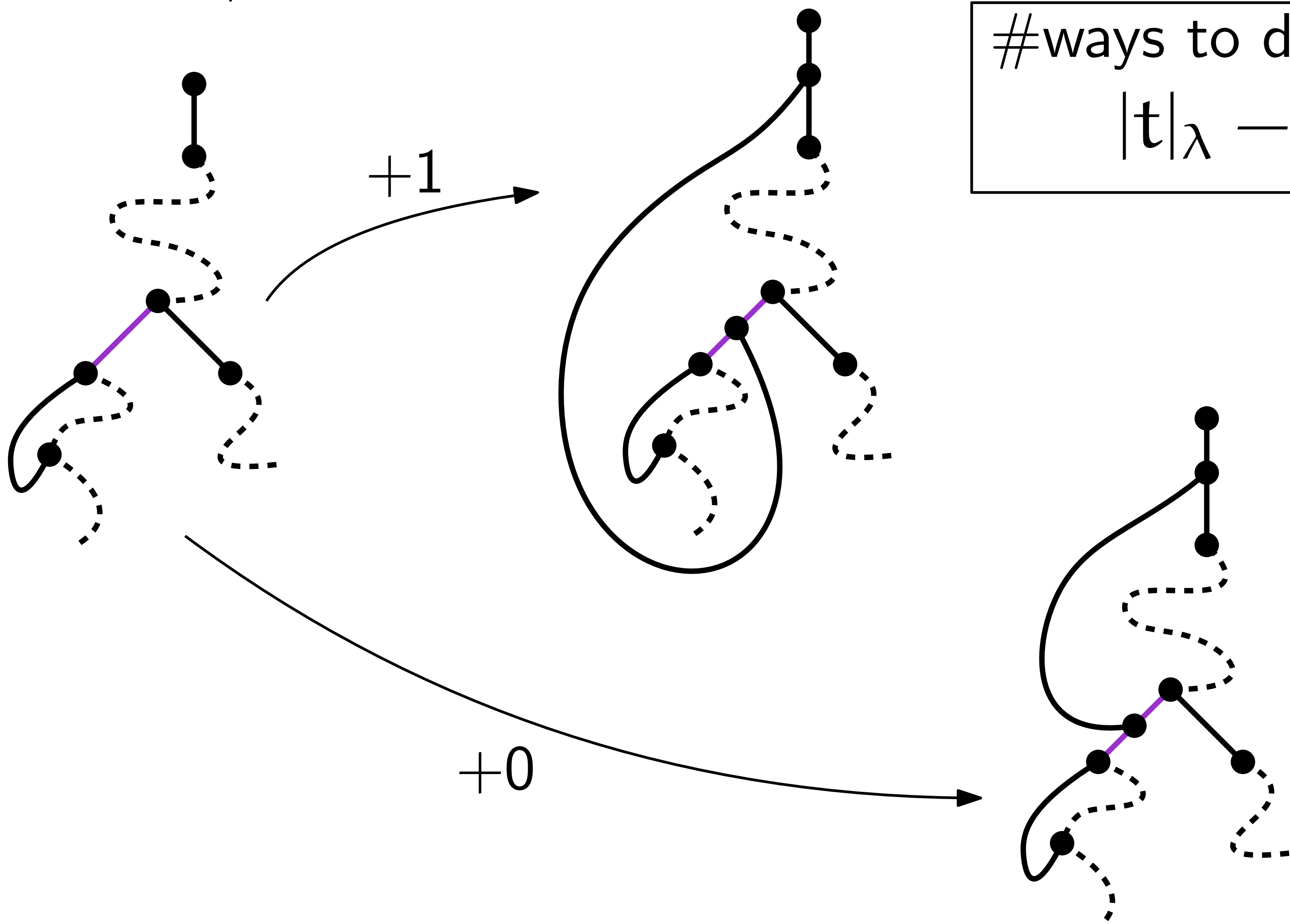
#ways to do this
$|t|_\beta$



Mean number of β-redices in closed terms (WIP)

- Tracking redices during the decomposition

Abstractions, subcase 1.2



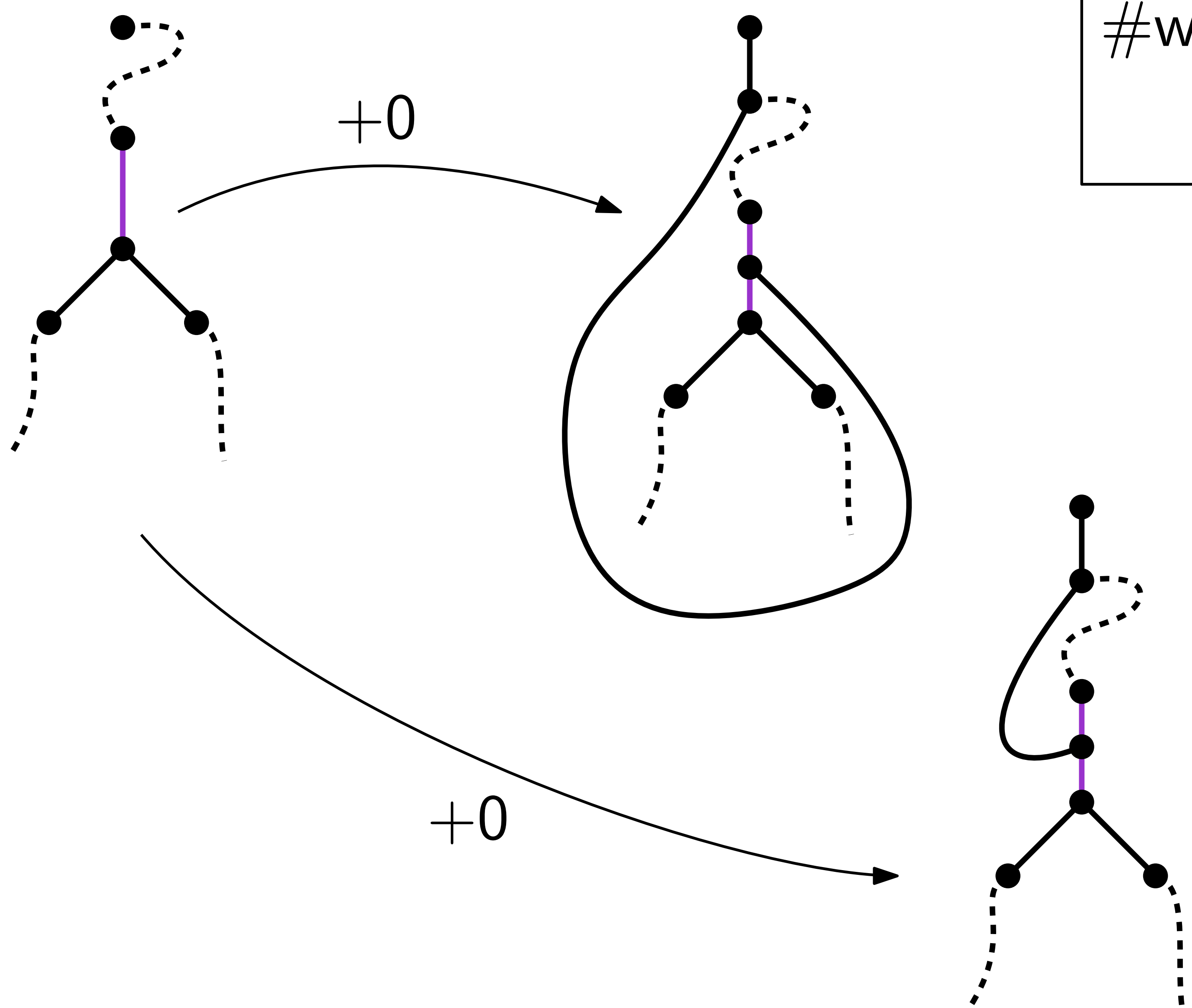
#ways to do this

$$|t|_\lambda - |t|_\beta$$

Mean number of β-redices in closed terms (WIP)

- Tracking redices during the decomposition

Abstractions, subcase 1.3



#ways to do this

$$|t| - |t|_\lambda$$

Mean number of β-redices in closed terms (WIP)

- Tracking redices during the decomposition
  - Using the following facts:
    - $n_\lambda = \frac{|t|+1}{3}, n_{\not\chi} = \frac{2|t|-1}{3}$
    - $r\partial_r T_0 = \sum_{t \in T_0} |t|_\beta z^{|t|} r^{|t|_\beta}$
    - $\frac{z\partial_z T_0 + T_0}{3} = \sum_{t \in T_0} \frac{|t|+1}{3} z^{|t|} v^{|t|_\beta}$
    - $\frac{2z\partial_z T_0 - T_0}{3} = \sum_{t \in T_0} \frac{2|t|-1}{3} z^{|t|} v^{|t|_\beta}$

Mean number of β-redices in closed terms (WIP)

- Translating to a diff-eq and pumping

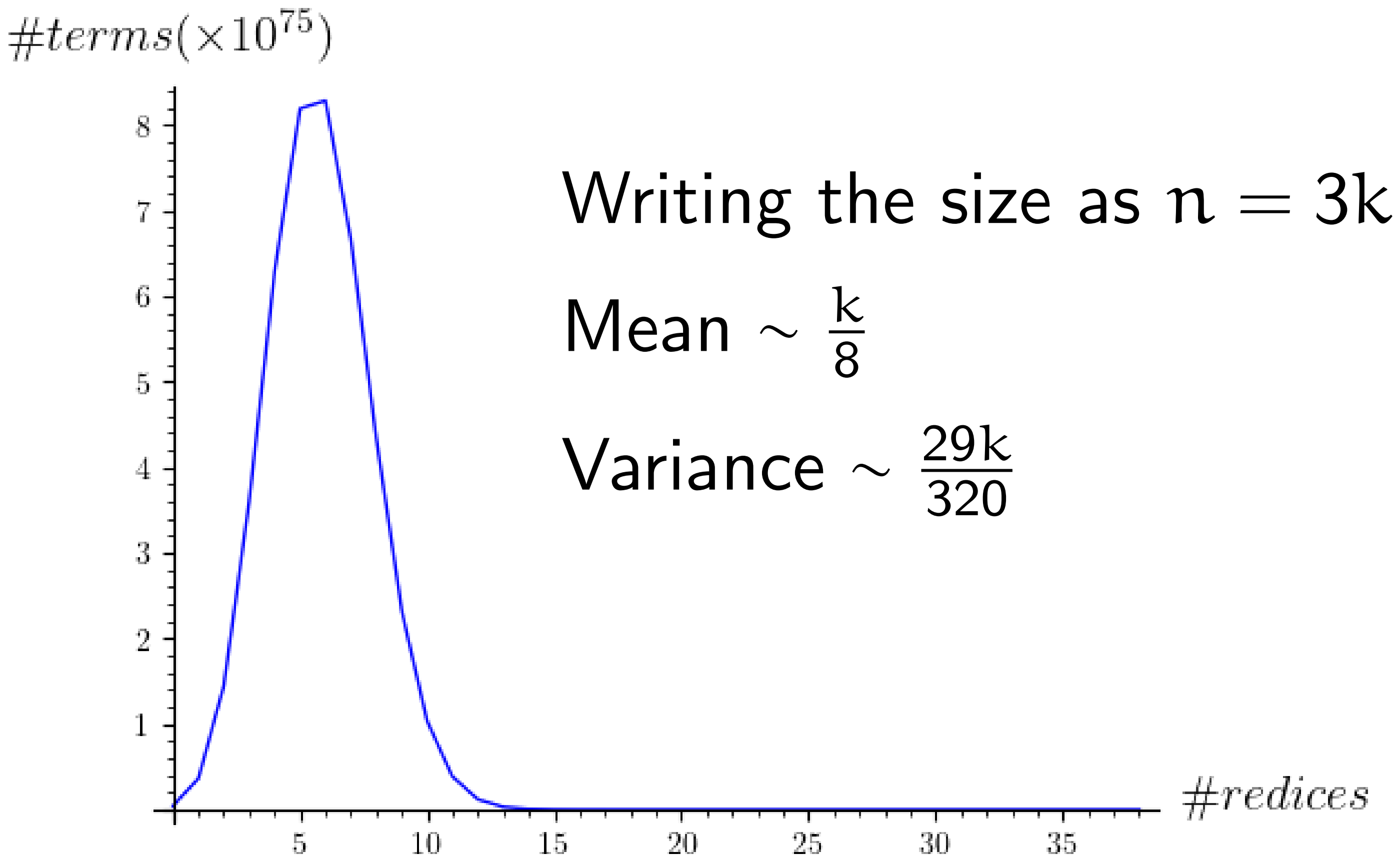
$$T_0 = -z\left(z^2(r+1)(1+(r-1)zT)(r-1)\partial_r T_0 - \frac{(1+z(r-1)T)z^3(r+5)\partial_z T_0}{3} - \frac{z^3(r-1)^2T_0^2}{3} - \frac{4z^2(r-1)T_0}{3} - z - T_0^2\right)$$

Mean number of β-redices in closed terms (WIP)

- Translating to a diff-eq and pumping

$$T_0 = -z\left(z^2(r+1)(1+(r-1)zT)(r-1)\partial_r T_0 - \frac{(1+z(r-1)T)z^3(r+5)\partial_z T_0}{3} - \frac{z^3(r-1)^2T_0^2}{3} - \frac{4z^2(r-1)T_0}{3} - z - T_0^2\right)$$

A plot of the dist. of redices for $n = 119$



Writing the size as $n = 3k + 2$, we have:

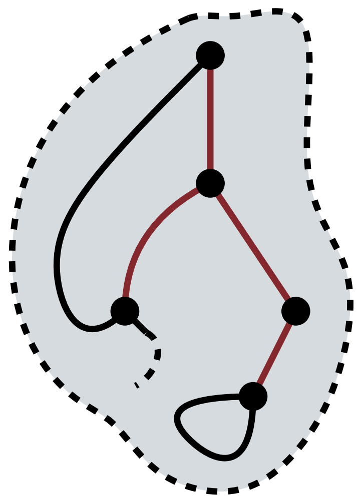
Mean $\sim \frac{k}{8}$

Variance $\sim \frac{29k}{320}$

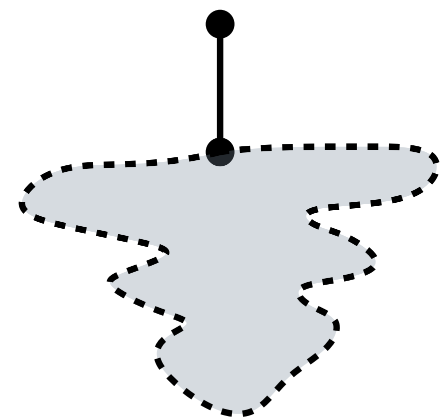
Whats next?

Whats next?

- More parameters:



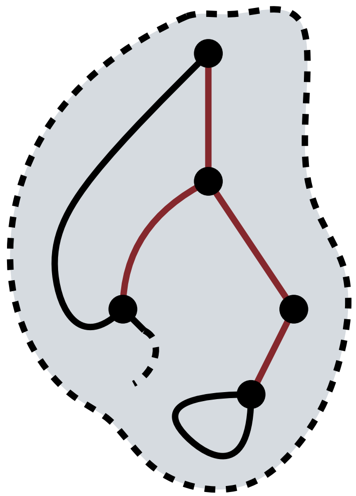
Mean path length



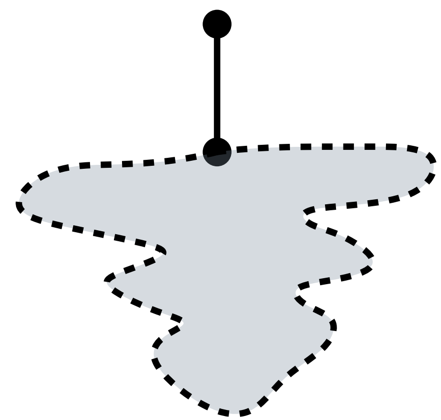
Profile

Whats next?

- More parameters:



Mean path length

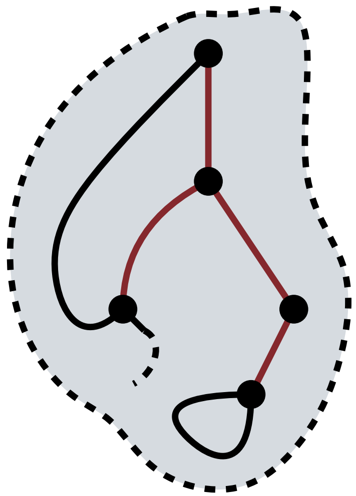


Profile

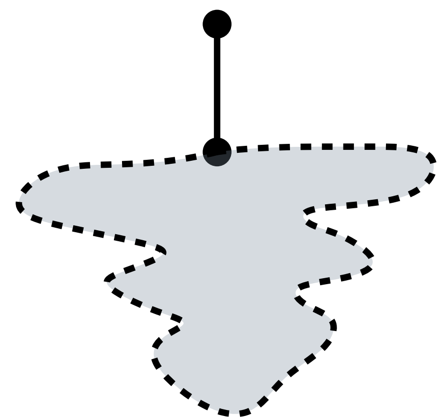
- More map/term families: planar, bridgeless...

Whats next?

- More parameters:



Mean path length



Profile

- More map/term families: planar, bridgeless...

Thank you!

Bibliography

[BGGJ13] Bodini, O., Gardy, D., Gittenberger, B., & Jacquot, A. (2013).
Enumeration of Generalized BCI Lambda-terms.
The Electronic Journal of Combinatorics, P30-P30.

[Z16] Zeilberger, N. (2016).
Linear lambda terms as invariants of rooted trivalent maps.
Journal of functional programming, 26.

[AB00] Arques, D., & Béraud, J. F. (2000).
Rooted maps on orientable surfaces, Riccati's equation and continued fractions
Discrete mathematics, 215(1-3), 1-12.

[BFSS01] Banderier, C., Flajolet, P., Schaeffer, G., & Soria, M. (2001).
Random maps, coalescing saddles, singularity analysis, and Airy phenomena.
Random Structures & Algorithms, 19(3-4), 194-246.

Bibliography

[BR86] Bender, E. A., & Richmond, L. B. (1986).
A survey of the asymptotic behaviour of maps.
Journal of Combinatorial Theory, Series B, 40(3), 297-329.

[BGLZ16] Bendkowski, M., Grygiel, K., Lescanne, P., & Zaionc, M. (2016).
A natural counting of lambda terms.
In International Conference on Current Trends in Theory and Practice of Informatics (pp. 183-194). Springer, Berlin, Heidelberg.

[BBD19] Bendkowski, M., Bodini, O., & Dovgal, S. (2019).
Statistical Properties of Lambda Terms.
The Electronic Journal of Combinatorics, P4-1.

[BCDH18] Bodini, O., Courtiel, J., Dovgal, S., & Hwang, H. K. (2018, June).
Asymptotic distribution of parameters in random maps.
In 29th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms (Vol. 110, pp. 13-1)

Bibliography

[B75] Bender, E. A. (1975).
An asymptotic expansion for the coefficients of some formal power series.
Journal of the London Mathematical Society, 2(3), 451-458.

[FS93] Flajolet, P., & Soria, M. (1993).
General combinatorial schemas: Gaussian limit distributions and exponential tails.
Discrete Mathematics, 114(1-3), 159-180.

[B18] Borinsky, M. (2018).
Generating Asymptotics for Factorially Divergent Sequences.
The Electronic Journal of Combinatorics, P4-1.

[BKW21] Banderier, C., Kuba, M., & Wallner, M. (2021).
Analytic Combinatorics of Composition schemes and phase transitions
mixed Poisson distributions.
arXiv preprint arXiv:2103.03751.

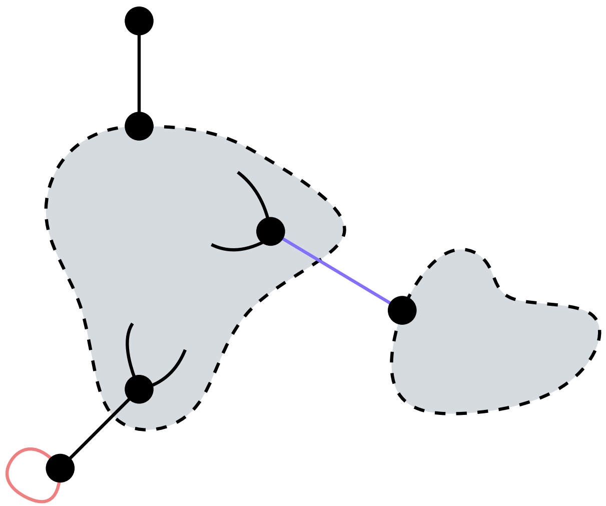
Bibliography

[P19] Panafieu, É. (2019).
Analytic combinatorics of connected graphs.
Random Structures & Algorithms, 55(2), 427-495.

[BGJ13] Bodini, O., Gardy, D., & Jacquot, A. (2013).
Asymptotics and random sampling for BCI and BCK lambda terms
Theoretical Computer Science, 502, 227-238.

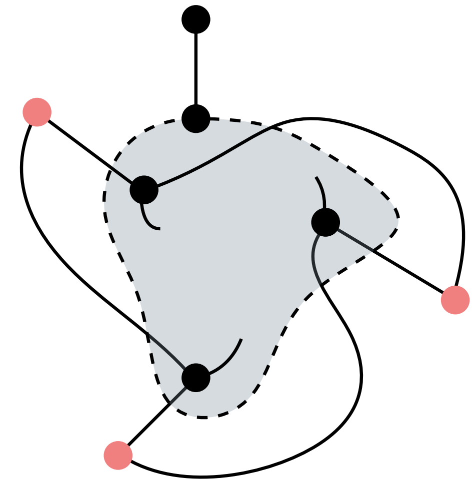
Our results: limit distributions

Trivalent maps ↔ closed linear terms



\# loops = #id-subterms
\# bridges = # closed subt.
} Poisson(1)

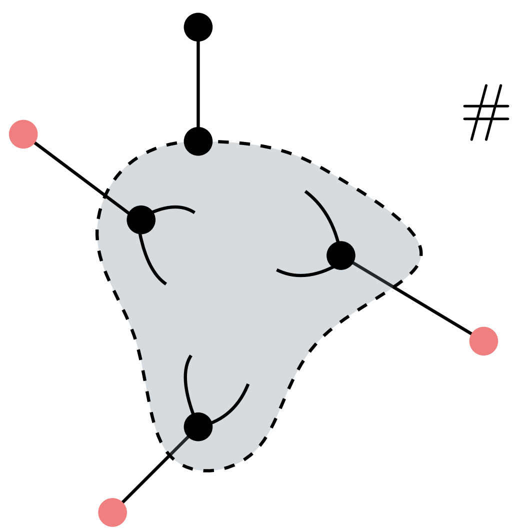
(2,3)-maps ↔ closed affine terms



\# unary vertices = # free vars

$\mathcal{N}(mu, \sigma^2)$ with $\mu = \sigma^2 = (2n)^{2/3}$

(1,3)-maps ↔ open linear terms



\# unary vertices = # free vars

$\mathcal{N}(\mu, \sigma^2)$ with

$\mu = \sigma^2 = (2n)^{1/3}$