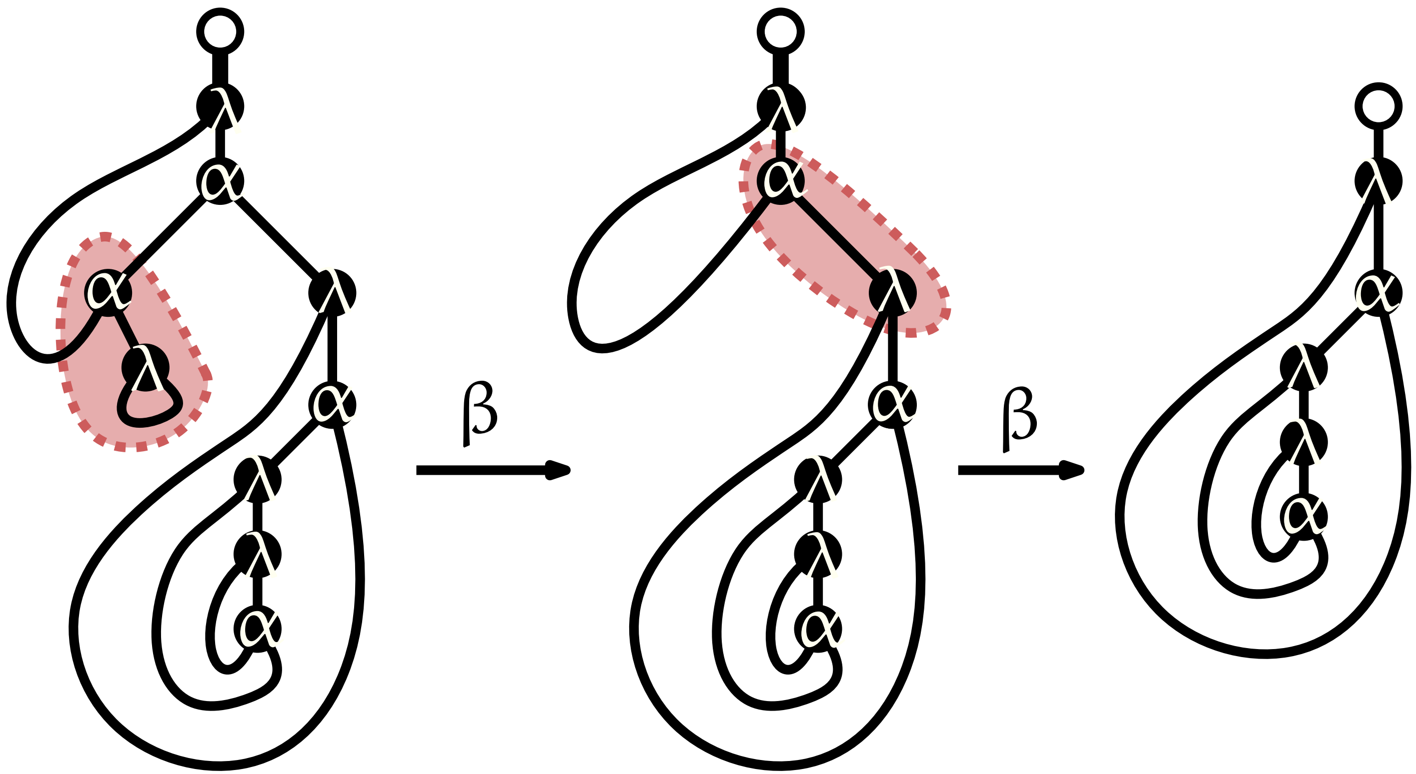
Normalisation of closed linear λ-terms and patterns in trivalent maps



Alexandros Singh

Based on joint work with Olivier Bodini, Bernhard Gittenberger
Michael Wallner, and Noam Zeilberger.

16th workshop on Computational Logic and Applications
Friday, January 13th 2023

The plan

- A brief overview of maps and the λ-calculus
- Context and results
- A strategy for deriving such results
- Normalisation of closed linear terms
- Other patterns in terms and maps

Quiz

asymptotically →

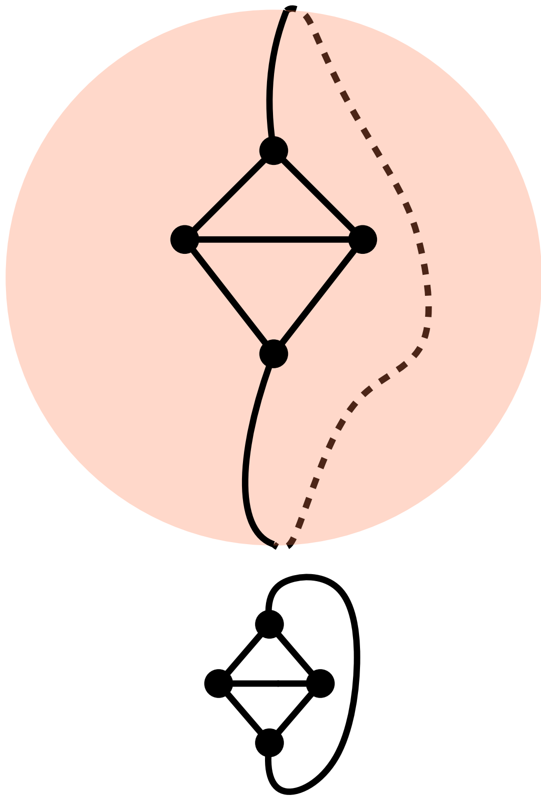
Of the following types of redices, which one has the highest mean number of occurences in random closed linear terms?

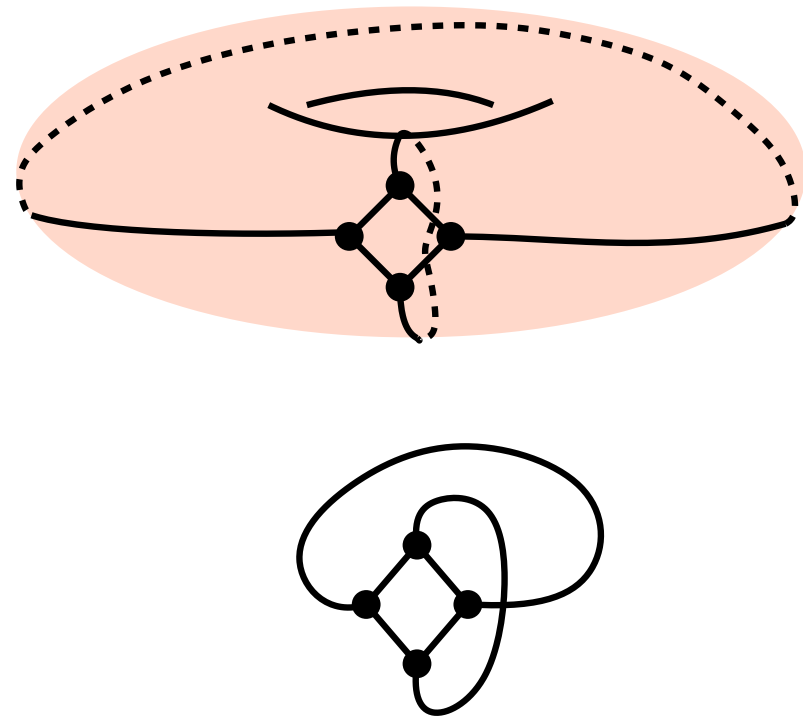
a) Abstraction applied to variable: $(\lambda x.t)\ y$

b) Abstraction applied to abstraction: $(\lambda x.t)\ (\lambda y.t')$

c) Abstraction applied to application: $(\lambda x.t)\ (a\ b)$

What are maps?





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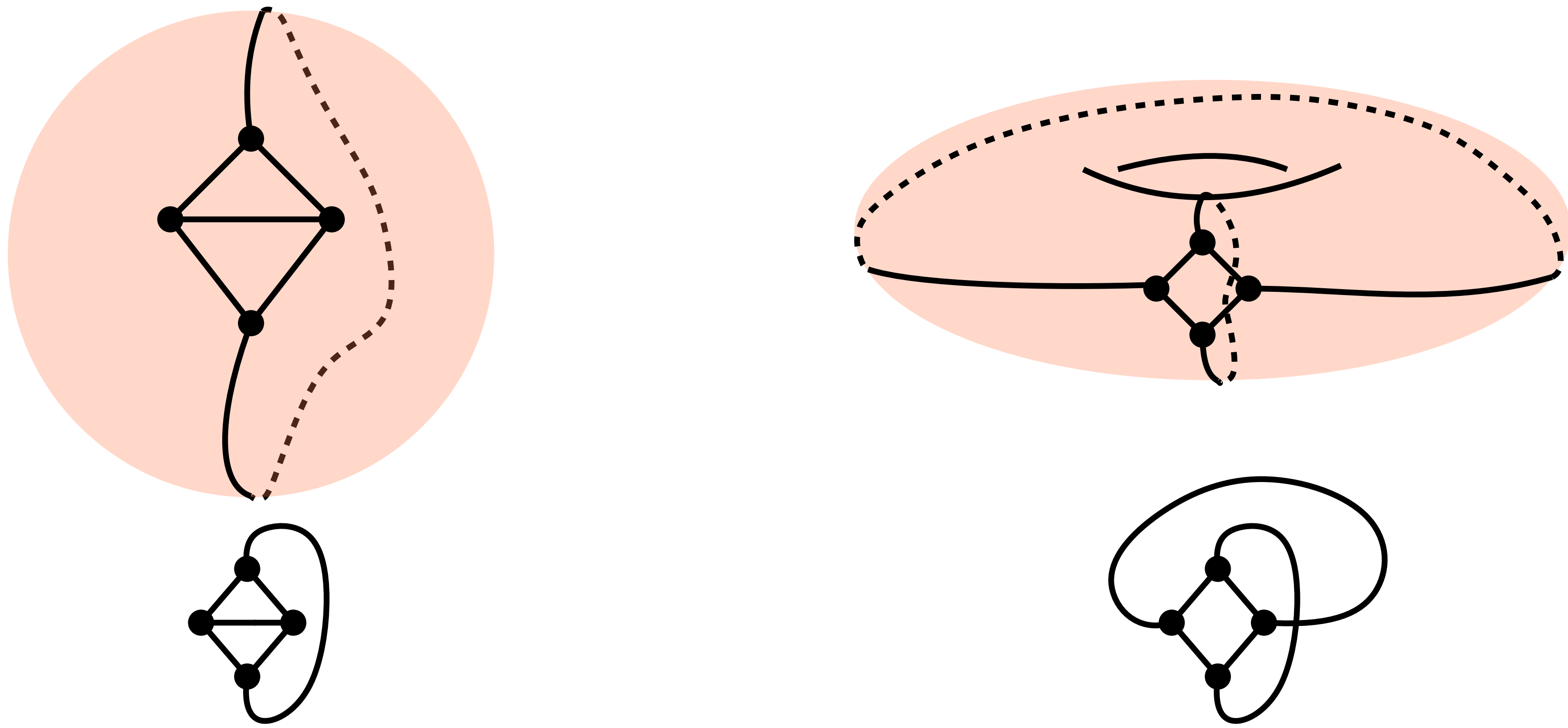
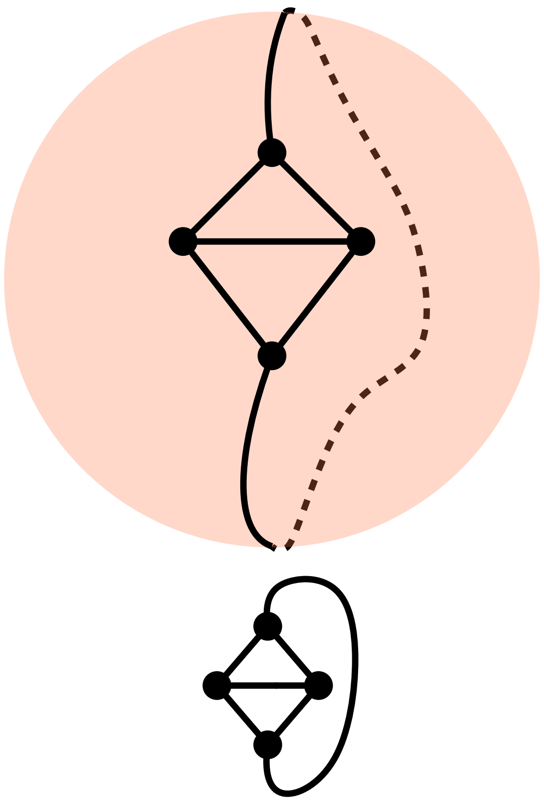


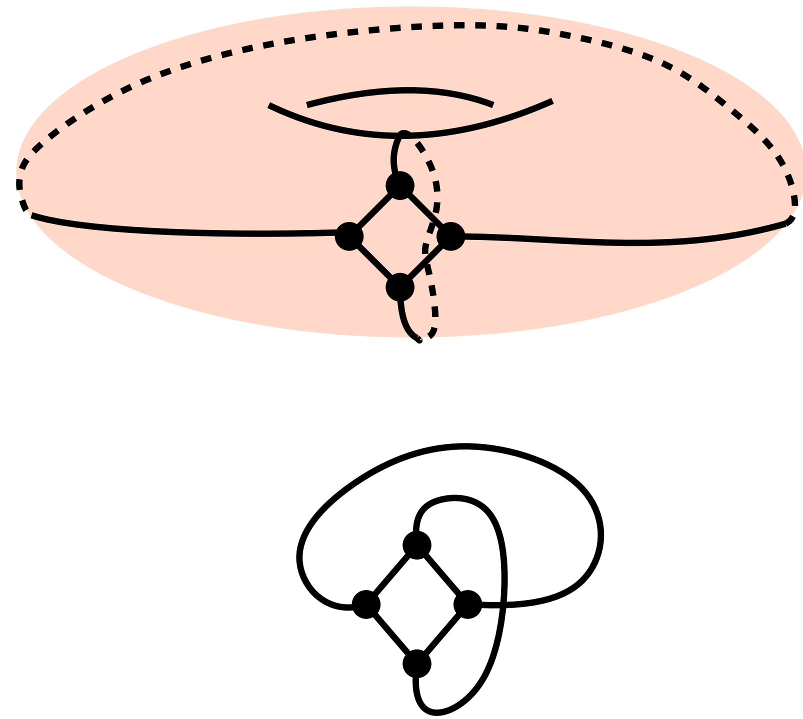
4CT...

- A central object in modern combinatorics, but not only that: probability, algebraic geometry, theoretical physics...

scaling limits... matrix integrals, Witten's conjecture, . . .

What are maps?





- A central object in modern combinatorics, but not only that: probability, algebraic geometry, theoretical physics...

- Their enumeration was pioneered by Tutte in the 60s, as part of his approach to the four colour theorem.

The untyped linear λ-calculus

- Its terms are formed inductively

$$\frac{}{x \vdash x}\ \text{var}$$

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Linking terms and maps

- In 2013, Bodini, Gardy, Jacquot, describe a series of bijections:

rooted trivalent maps $\leftrightarrow$ closed linear terms
rooted (2,3)-valent maps $\leftrightarrow$ closed affine terms

In the same year, together with Gittenberger, they study:

$\mathrm{BCI}(\mathrm{p})$ terms (each bound variable appears p times)

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- In 2015, Zeilberger advocates for

"linear lambda terms as invariants of rooted trivalent maps"

The thesis in context

Related work has been carried out on:

- Parameter studies on general λ-terms (ex., [BBD19]).

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Our focus is on:

- Trivalent maps, linear terms, and related families.
- Exploring the combinatorial interplay of maps and terms.
- Study of pairs of parameters on maps and terms.

Our results ●=w. Bodini, Zeilberger

Parameters on maps and terms of arbitrary genus (number of):

- Loops in trivalent maps and identity-subterms in closed linear terms

Limit law: $\mathrm{Poisson}(1)$

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Parameters on maps and terms of arbitrary genus (number of):

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- Bridges in trivalent maps and closed subterms in closed linear terms

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→ this talk!

Our strategy:

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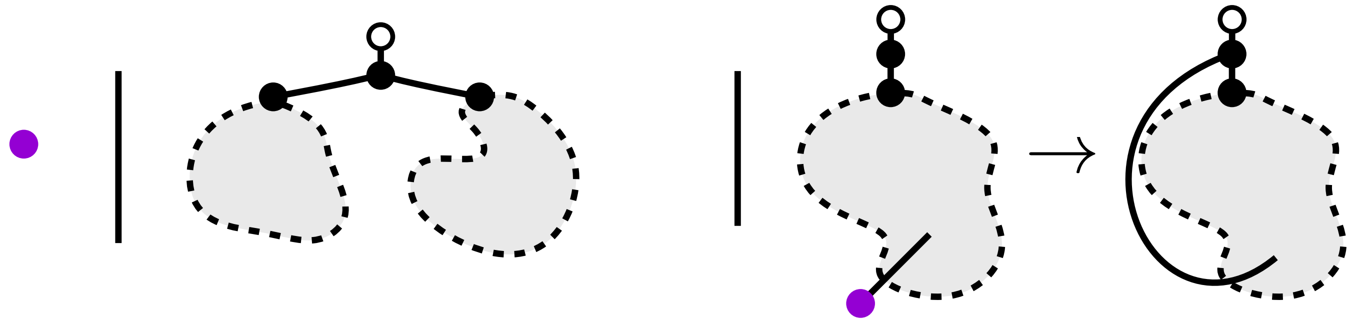
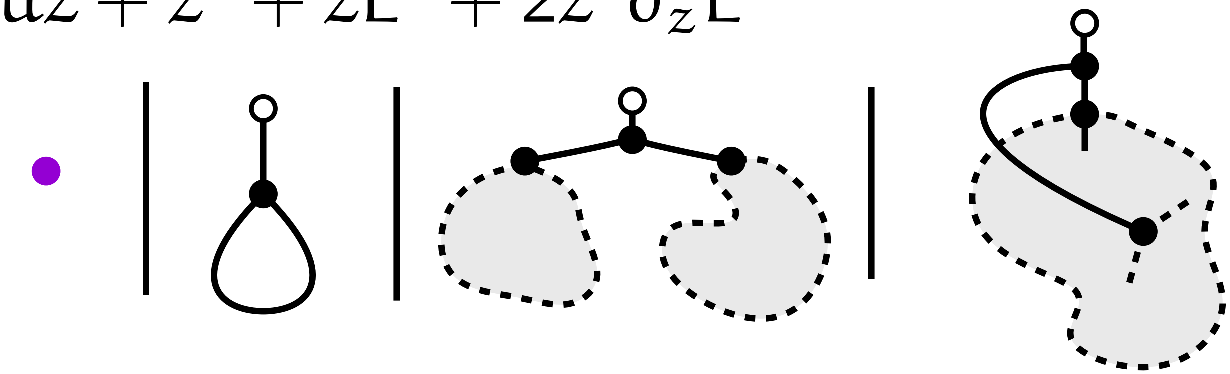
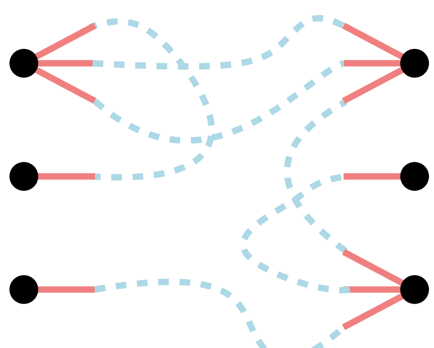


- $T = uz + z^2 + zL^2 + 2z^4\partial_z L$



- $T = uz^2 + z^4 + z^5 \frac{\partial}{\partial z}\left(\ln\left(\exp(z^2/2) \odot \exp(z^3/3 + uz)\right)\right)$



Our strategy:

1) Track evolution of parameters in decompositions of maps/λ-terms

different decompositions $\rightsquigarrow$ differential equations, Hadamard products, ...

generating functions divergent away from 0

2) Develop tools for rapidly growing coefficients, based on:

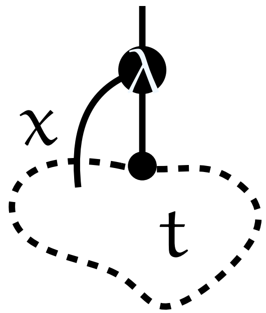
- Moment pumping
- Bender's theorem for compositions $F(z, G(z))$
- Coefficient asymptotics of Cauchy products

$$[z^n](A(z) \cdot B(z)) \sim a_n b_0 + a_0 b_n + O(a_{n-1} + b_{n-1})$$

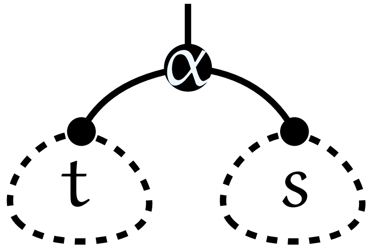
for A, B, G divergent and F analytic

From closed terms to maps

$\lambda x.t \to$



$(s\ t) \to$



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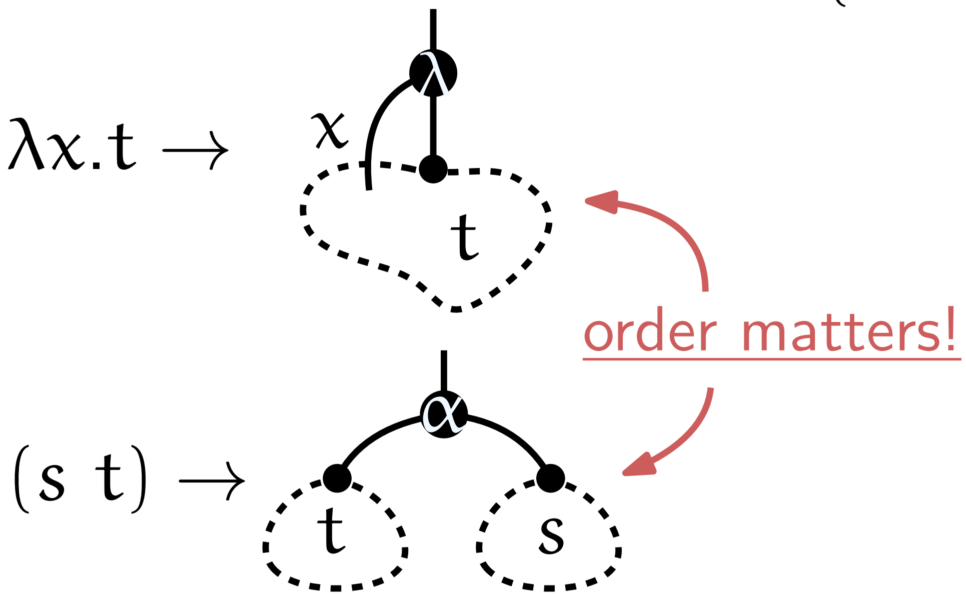
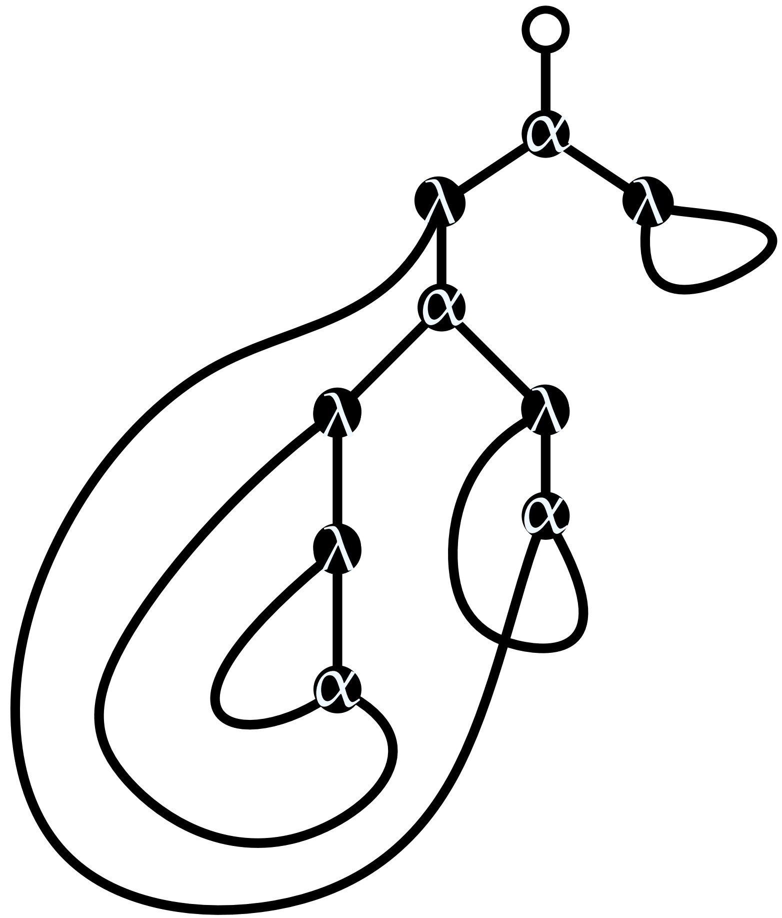


$(\lambda x.x)\ (\lambda y.(\lambda z.z\ y)\ (\lambda w.\lambda u.w\ u))$



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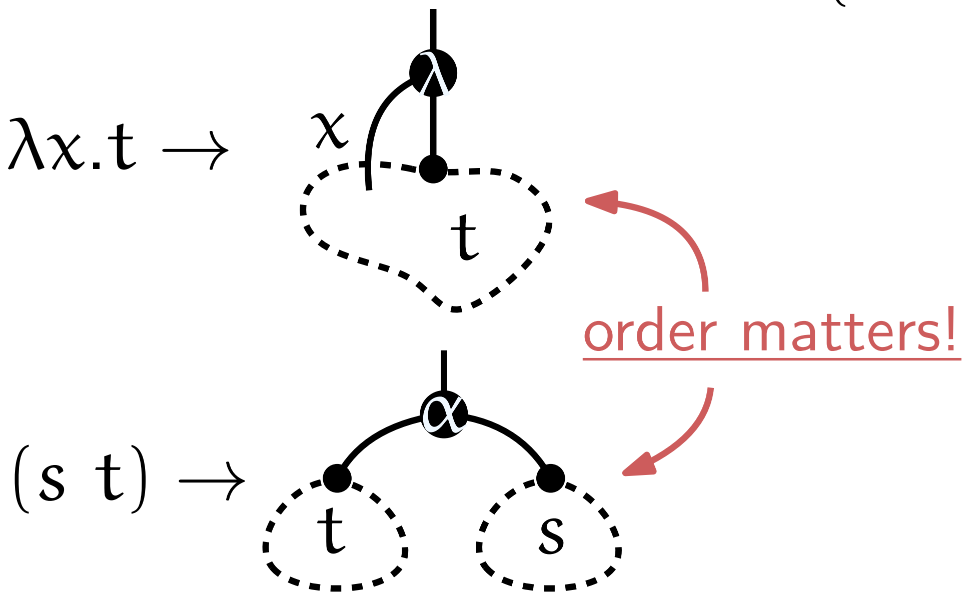

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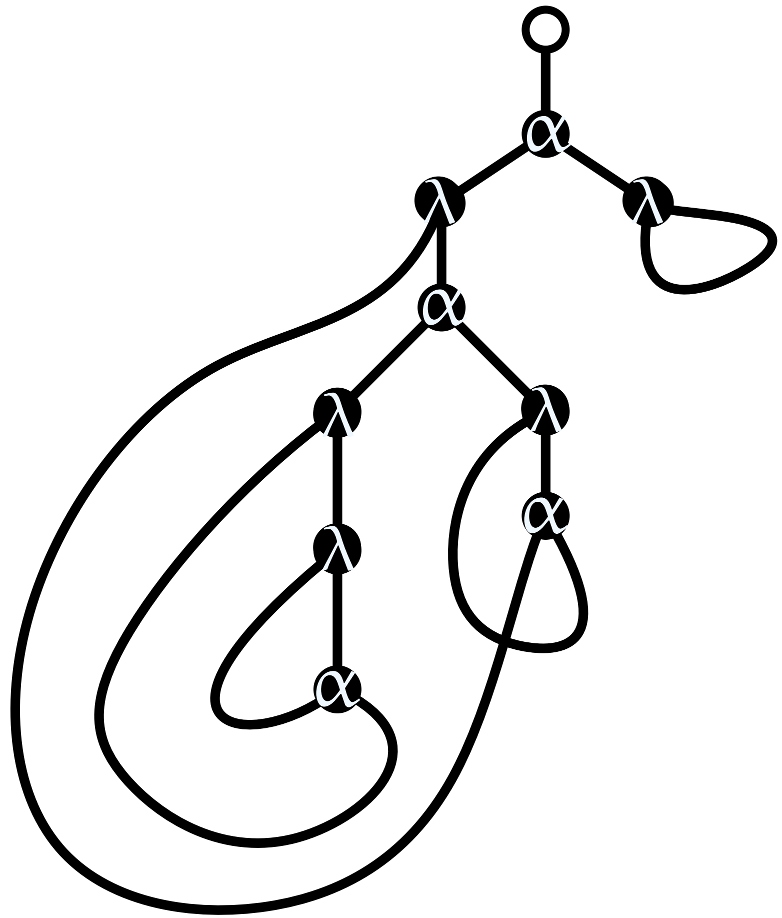
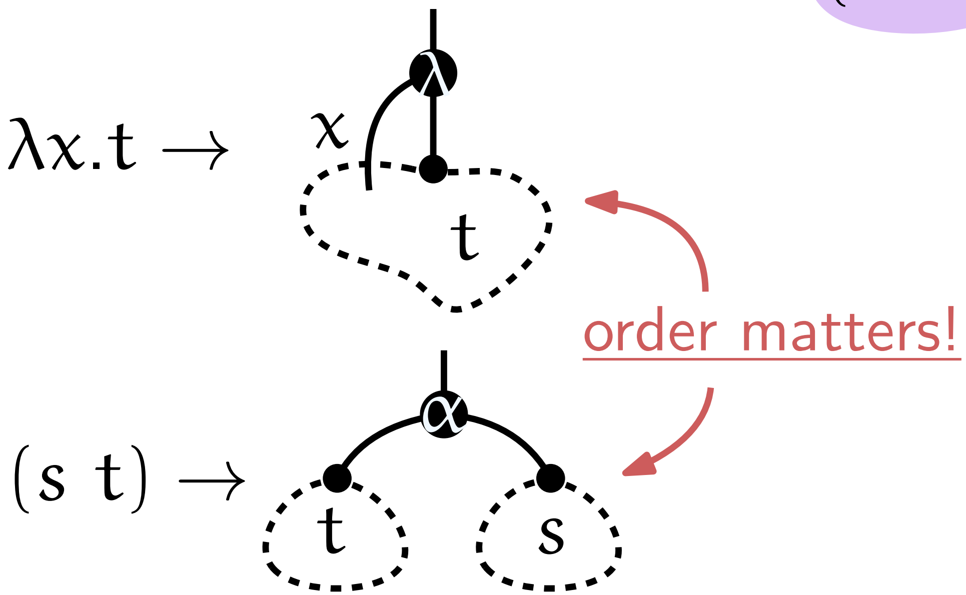
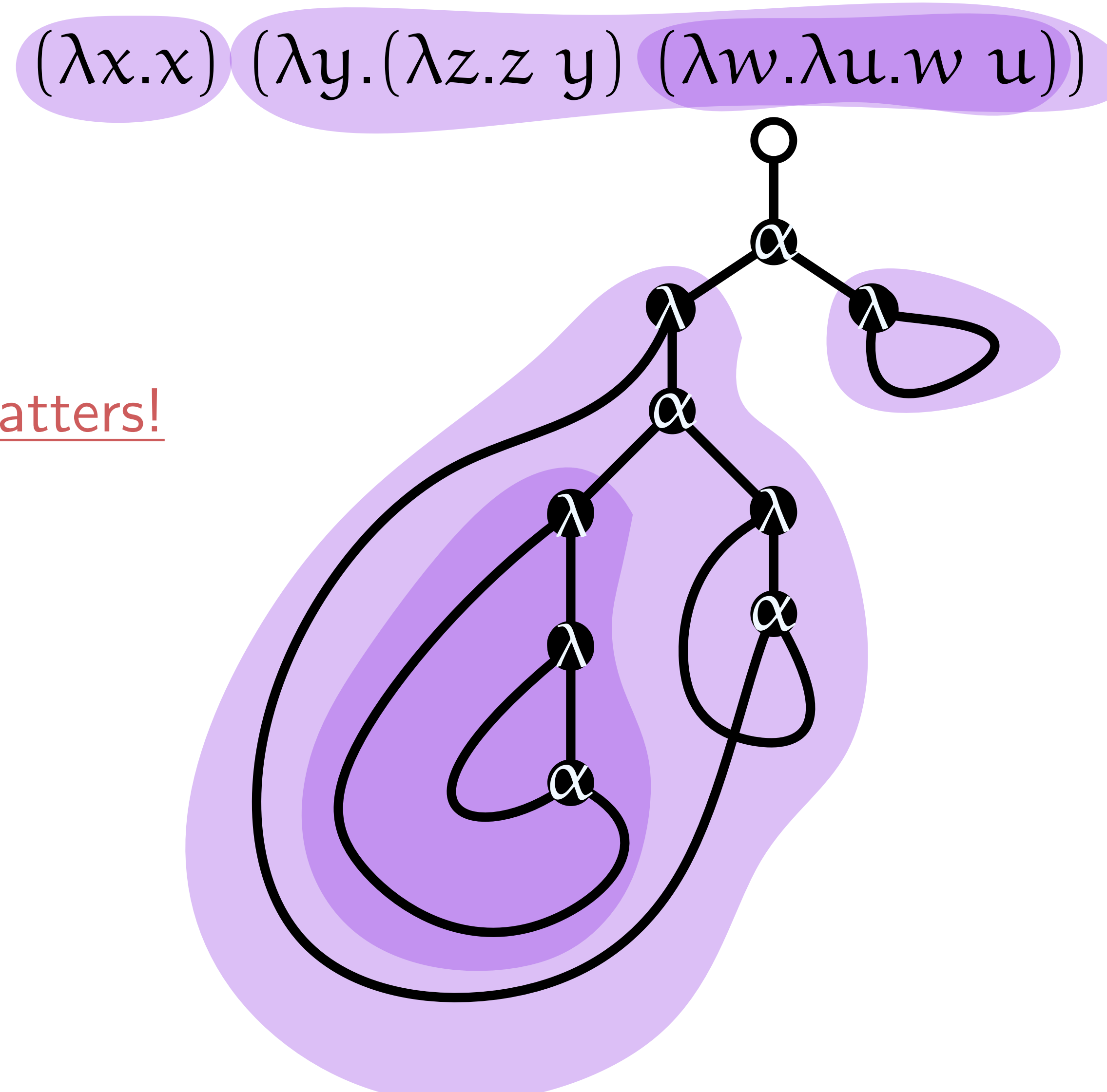


Dictionary

- # subterms ↔ # edges

From closed terms to maps





Dictionary

- # subterms ↔ # edges
- closed subterms ↔ bridges

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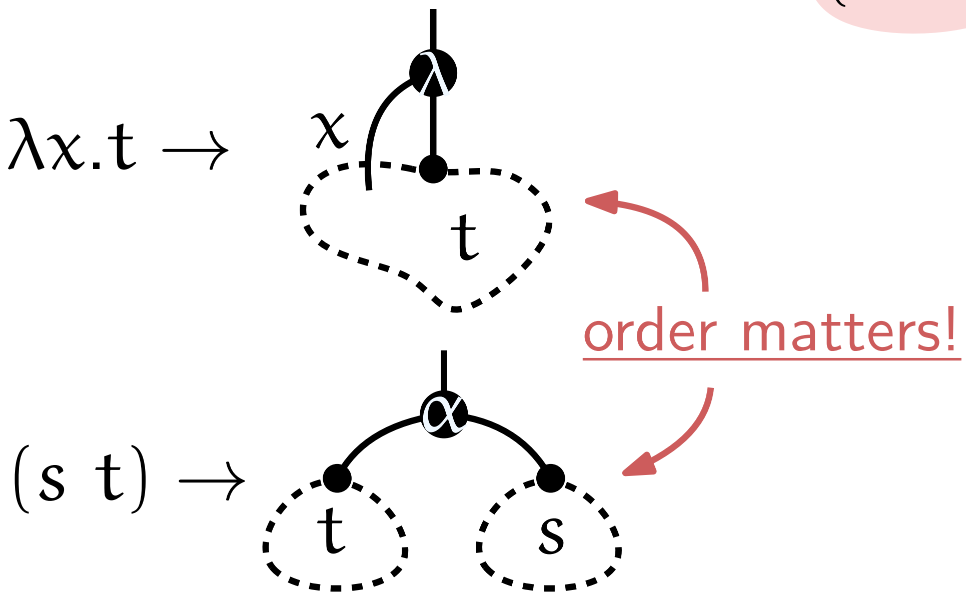

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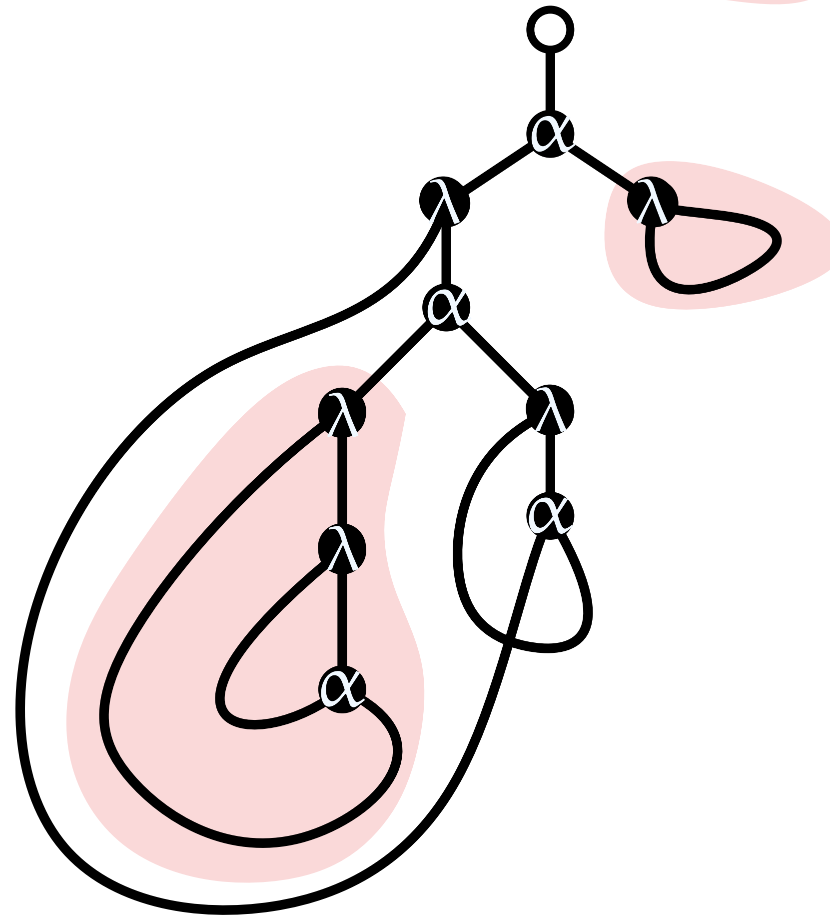

Dictionary

- $\#$ subterms $\leftrightarrow$ $\#$ edges
- closed subterms $\leftrightarrow$ bridges
- using variables in order $\leftrightarrow$ planarity of maps

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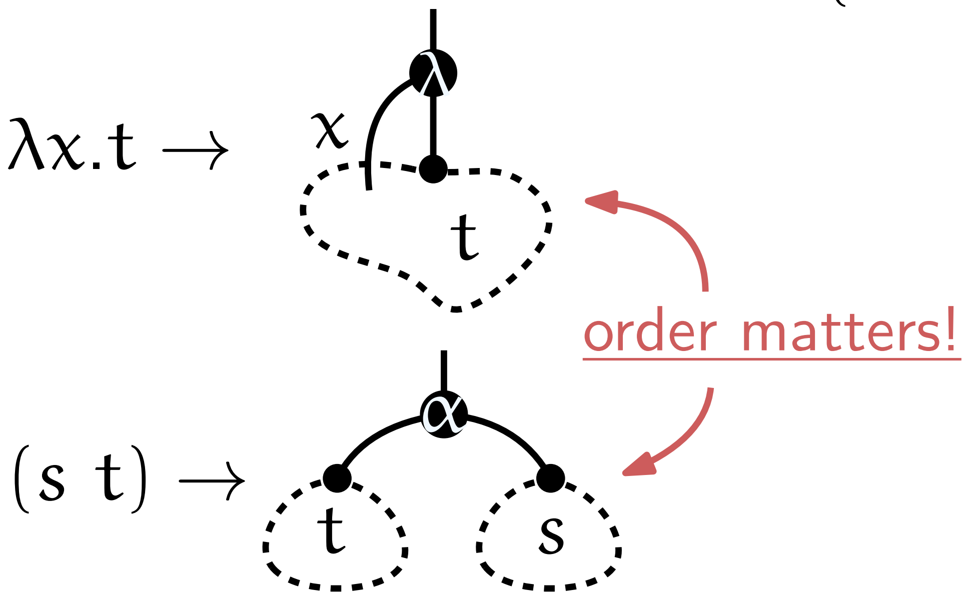

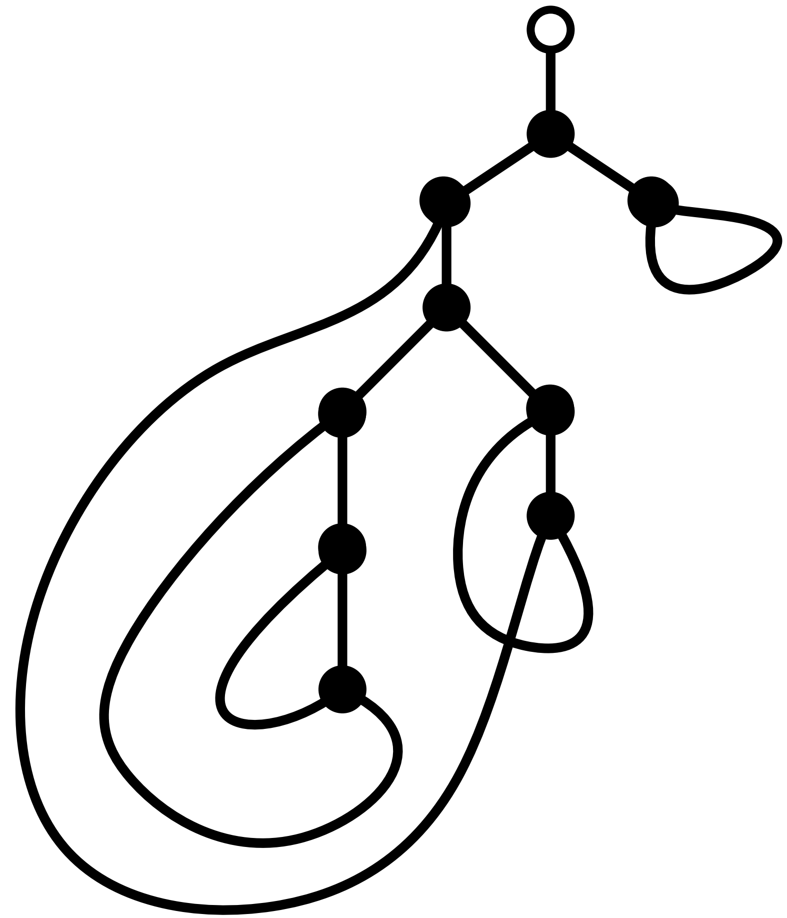

Dictionary

- # subterms ↔ # edges
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Q: What if we erase the labels? Can we recover them?

A: Yes, via an exploration process! [BGJ13, BGGJ13,Z16]

Decomposing rooted trivalent maps

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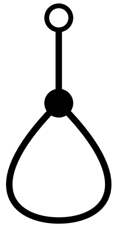

edges

$$T(z) = z^2$$

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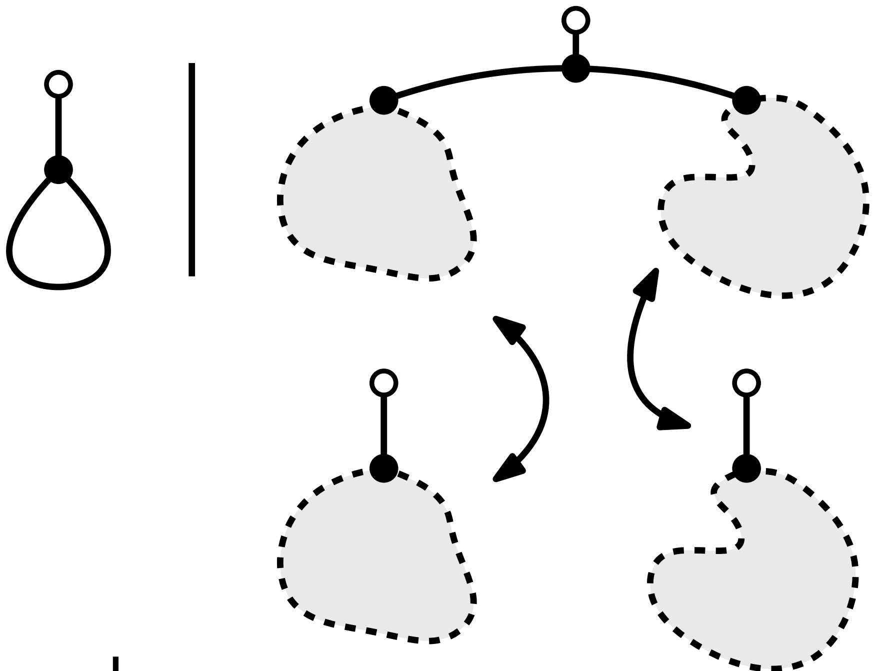

edges

$$T(z) = z^2 + zT(z)^2$$

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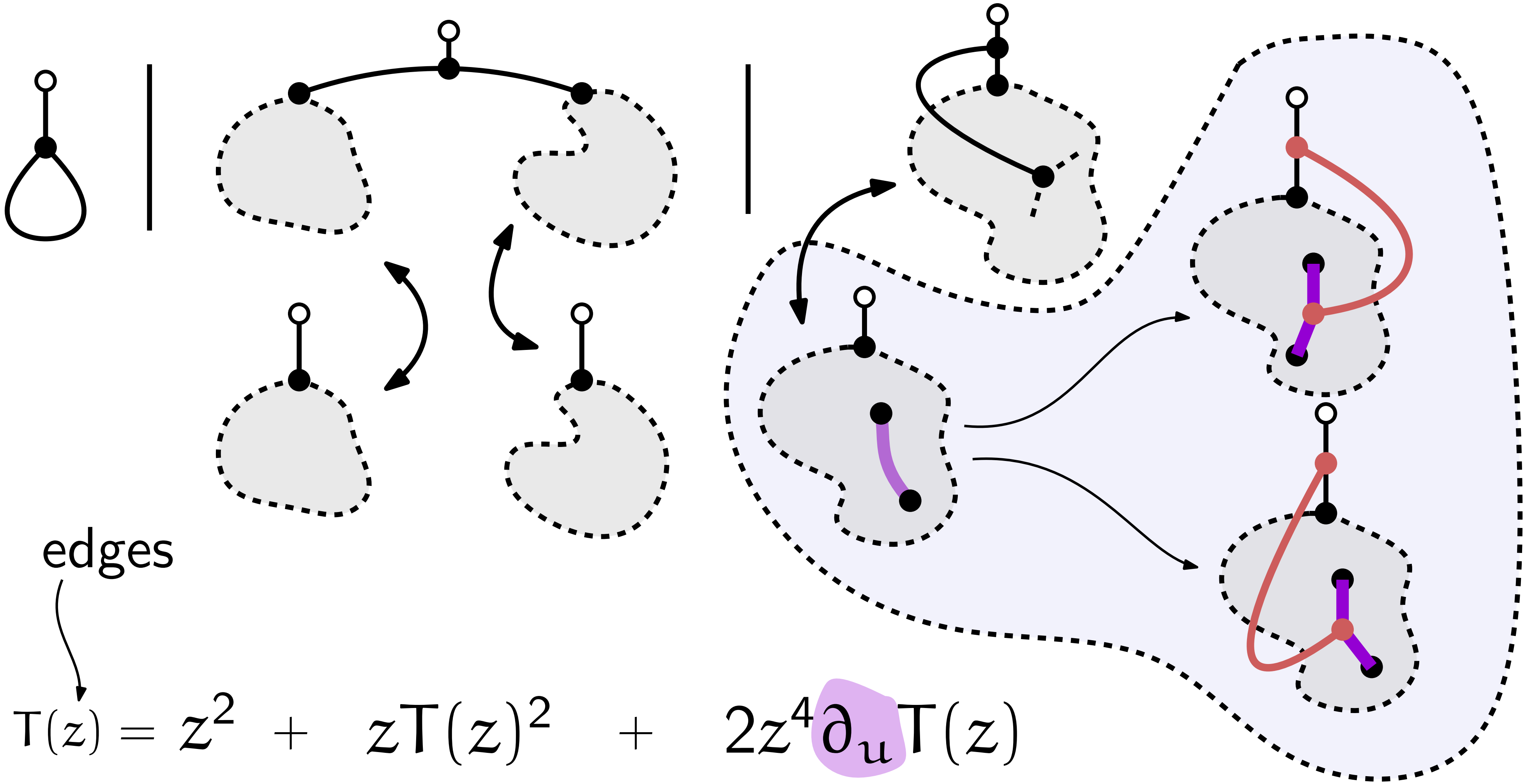
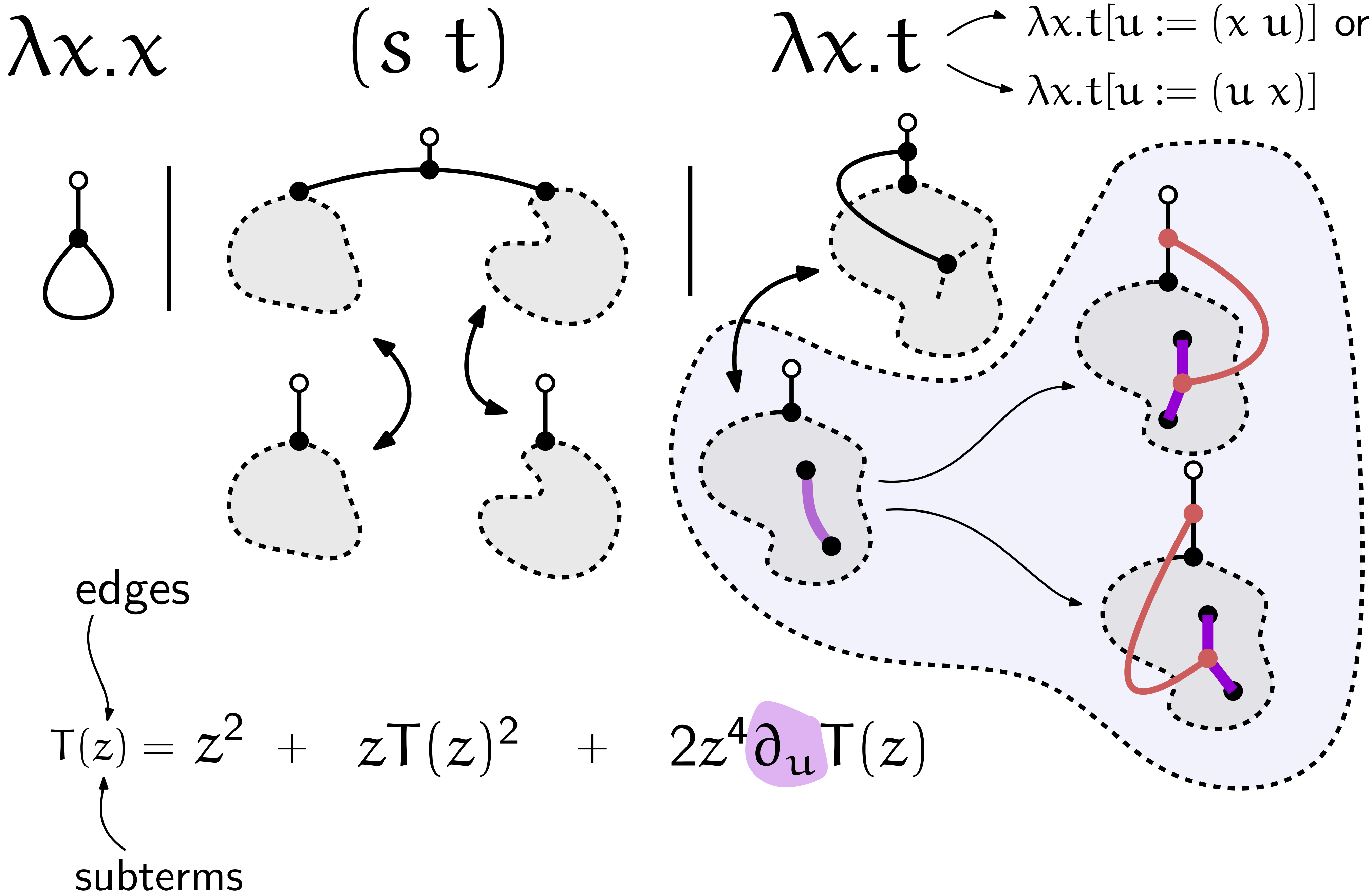


$$T(z) = z^2 + zT(z)^2 + 2z^4 \partial_u T(z)$$

Decomposing rooted trivalent maps and closed linear terms!



Computing with the λ-calculus

Dynamics of the λ-calculus: β-reductions

$$((\lambda x.t_1)\ t_2) \xrightarrow{\beta} t_1[x := t_2]$$

represents:

$$f = x \mapsto t_1$$

$f(t_2)$: replace x with t_2 inside t_1

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Examples of reductions

$$((\lambda x.x)\ y) \xrightarrow{\beta} x[x := y] = y$$

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$$((\lambda x.((\lambda y.(y\ x))\ z))\ (a\ b)) \xrightarrow{\beta} (\lambda x.(z\ x))(a\ b) \xrightarrow{\beta} (z(a\ b))$$

A term with no redices is called a normal form

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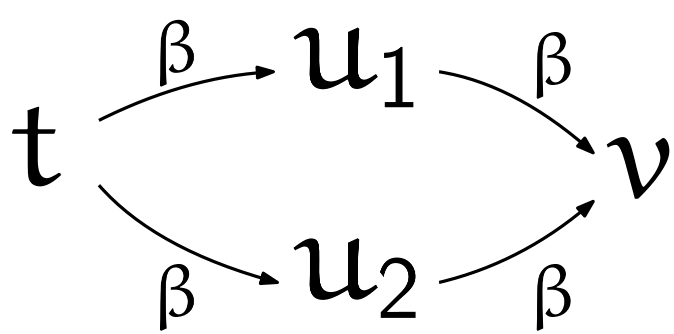
For linear terms: β-reduction is strongly normalising, has strong diamond property.



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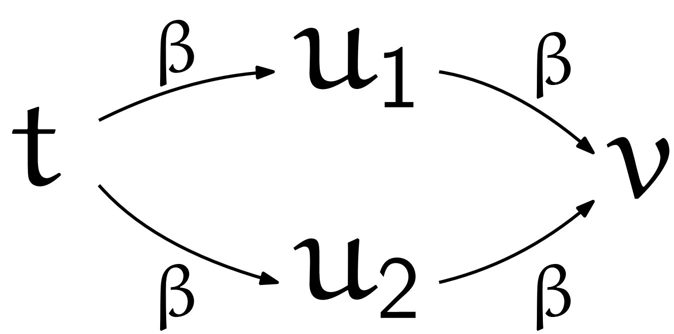
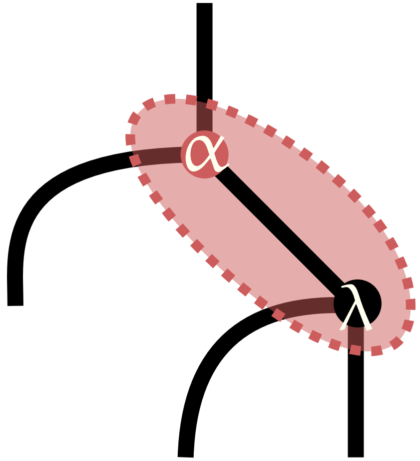


β-normalisation terminates in deterministic number of steps

β-reduction in maps



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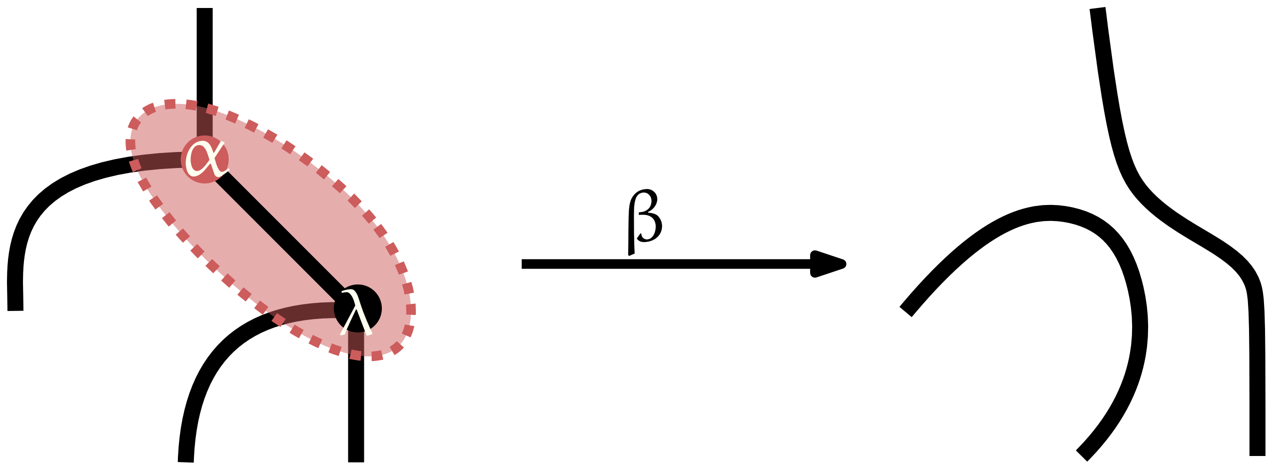

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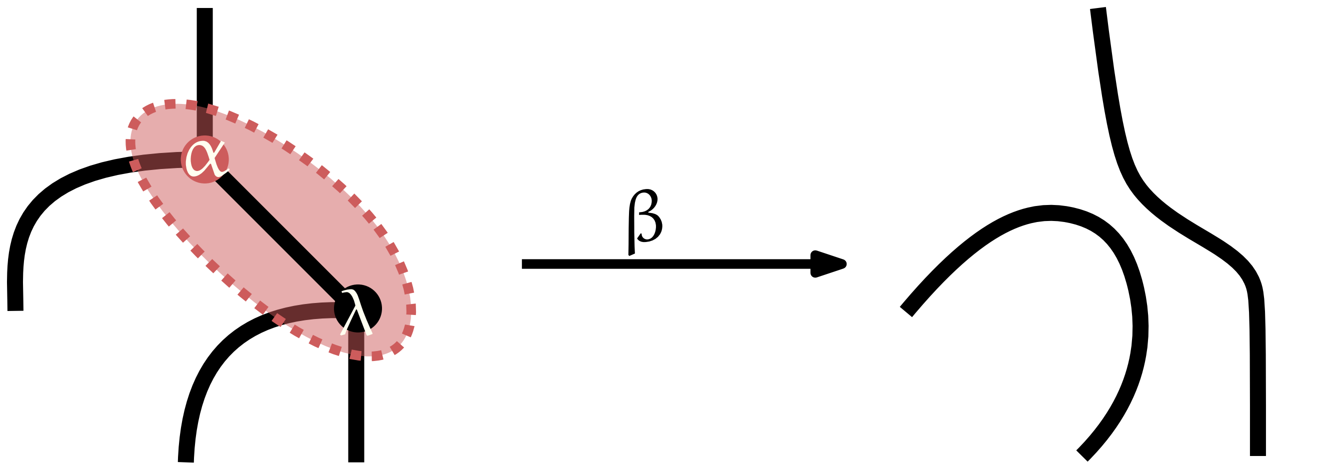

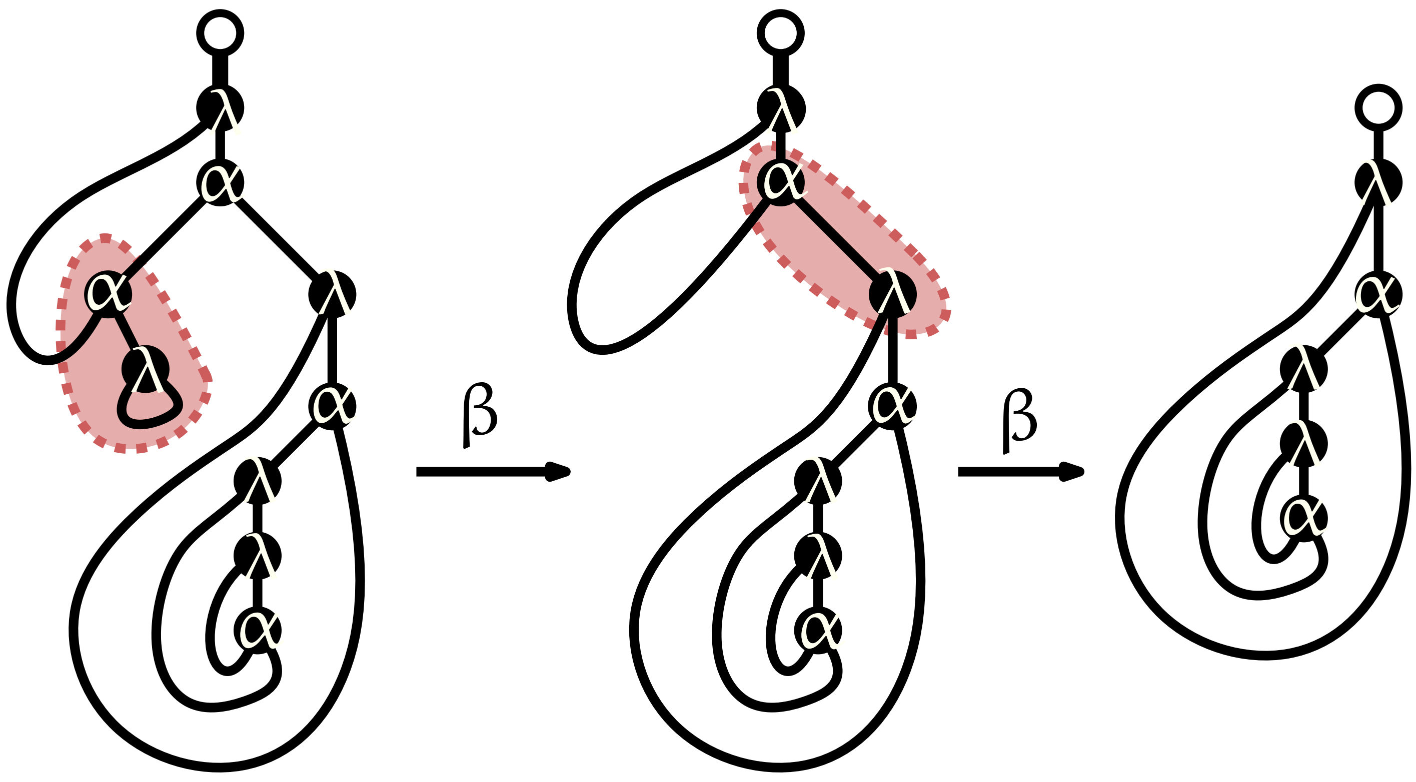

Normalisation of random closed linear terms

well defined! (strong normalisation + diamond)

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Q: Why is this just a lower bound?

A: Because reducing a redex can create a new one!

Mean number of β-redices in closed terms

Tracking redices: starts off easy...

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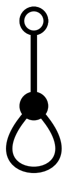
loops



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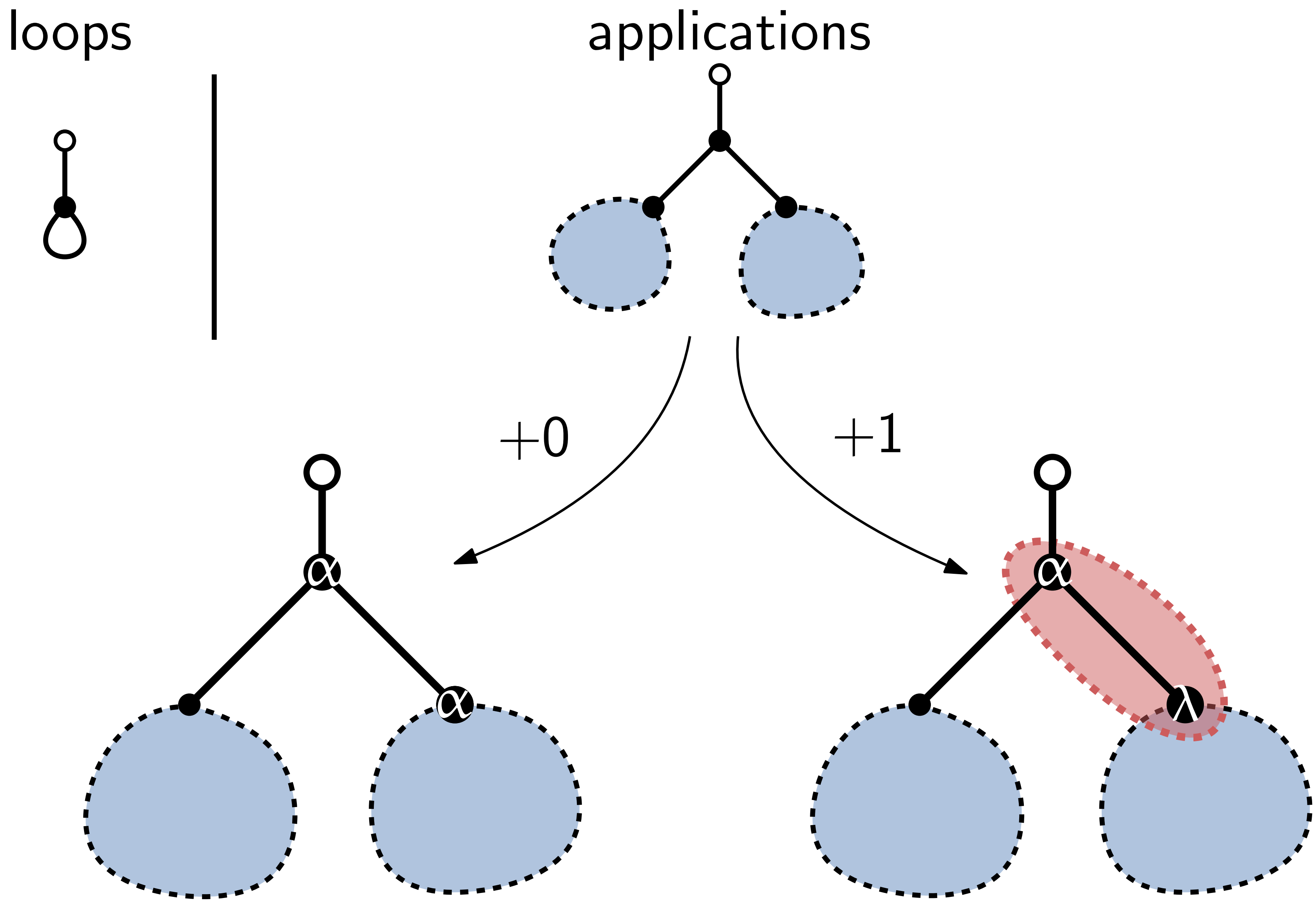
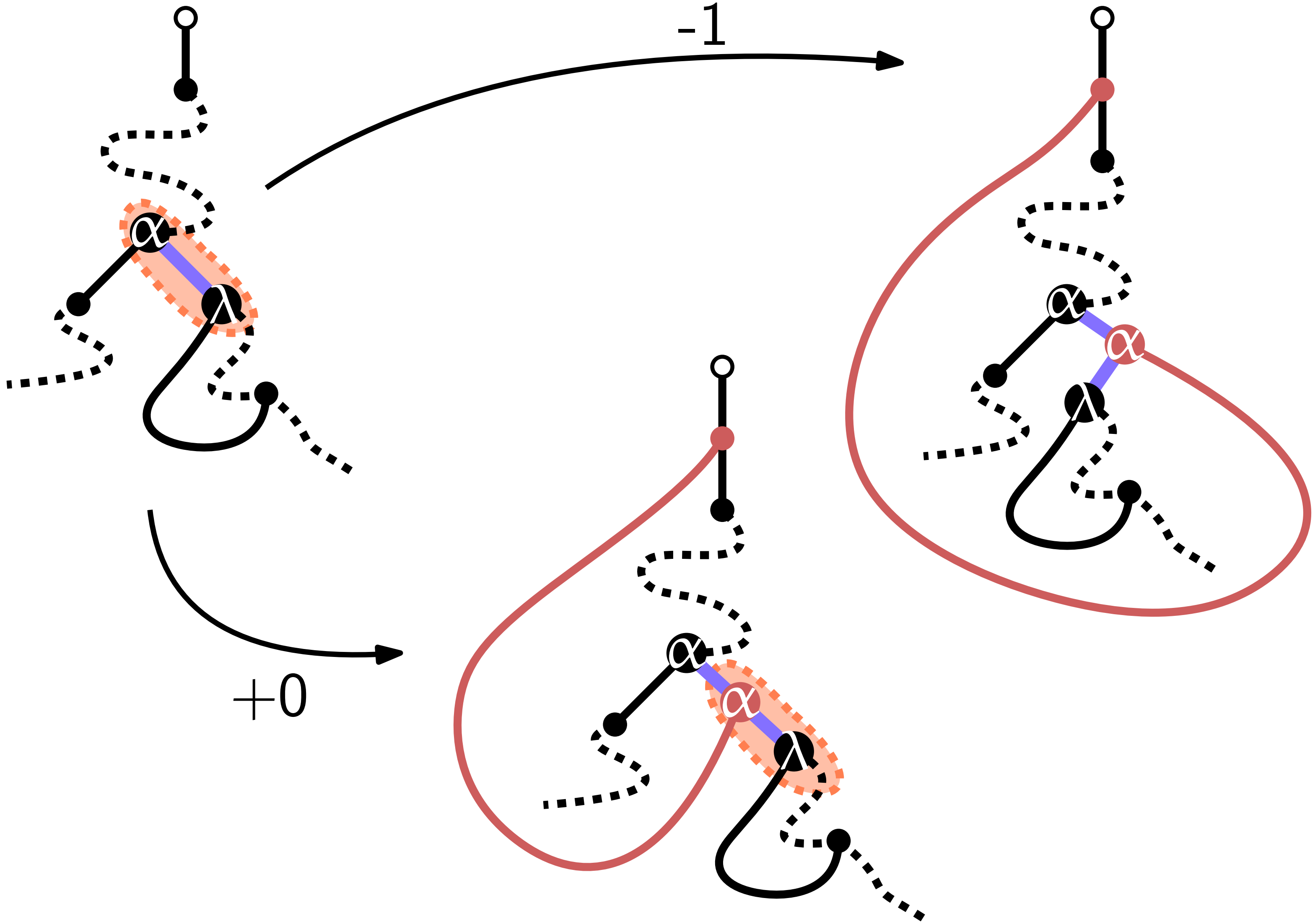


Mean number of β-redices in closed terms

Tracking redices: then gets harder!

Abstractions, subcase 1.1



Mean number of β-redices in closed terms

Translating to a differential equation and pumping

$$T = z^2 + zT^2 + z^3(1 + (r-1)zT)\left(\frac{z(r+5)\partial_z T}{3} - (r^2-1)\partial_r T)\right) + \frac{z^4(r-1)^2T^2}{3} + \frac{4z^3(r-1)T}{3}$$

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Let X_n be the random variable given by number of redices in a closed linear term of size $n \in 3\mathbb{N} + 2$. Then

$$\mathbb{E}(X_n) \sim \frac{n}{24}$$

$$\mathbb{V}(X_n) \sim \frac{n}{24}$$

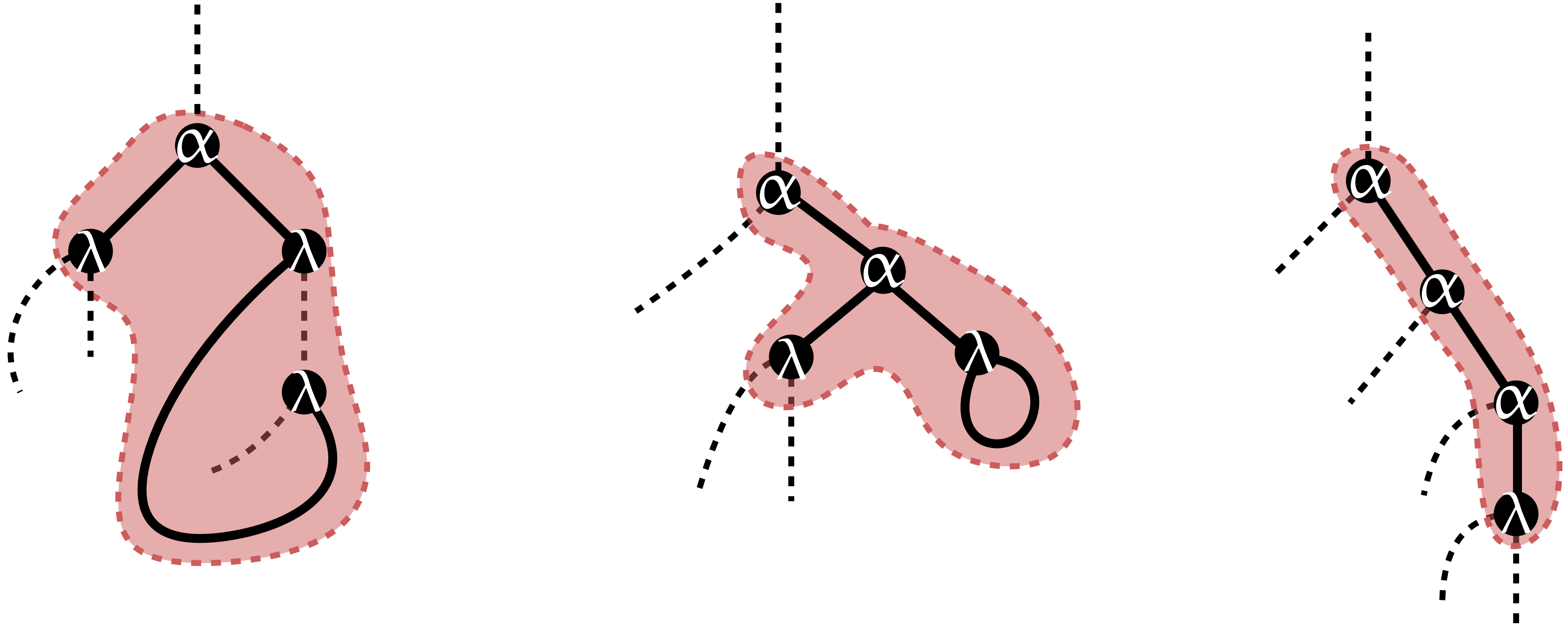
A lower bound for normalisation

Refining our counting to track reproducing redices:

A lower bound for normalisation

(see JJ Lévy's thesis)

Refining our counting to track reproducing redices:



$$p_1 = (\lambda x.C[(x\ u)])(\lambda y.t) \xrightarrow{\beta} C[((\lambda y.t)\ u)]$$

$$p_2 = (\lambda x.x)(\lambda y.t_1)t_2 \xrightarrow{\beta} (\lambda y.t_1)t_2$$

$$p_3 = ((\lambda x.\lambda y.t_1)\ t_2)\ t_3 \xrightarrow{\beta} (\lambda y.t_1[x := t_2])\ t_3$$

Enumerating $\mathfrak{p}_1$-patterns

- Tracking the creation/destruction of patterns during the recursive decomposition:

Enumerating p_1-patterns

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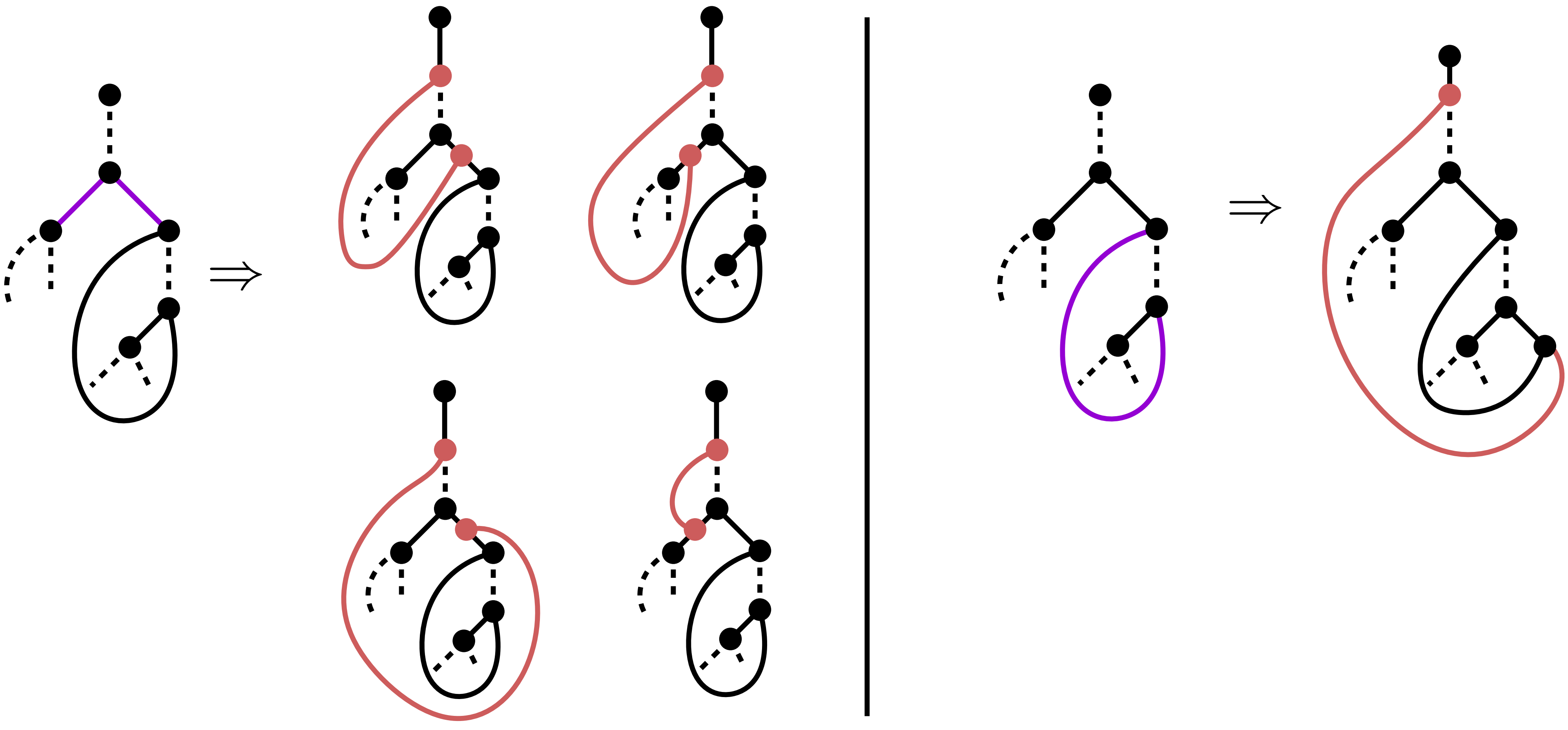
Cuts destroying a p_1-pattern:



Enumerating p_1-patterns

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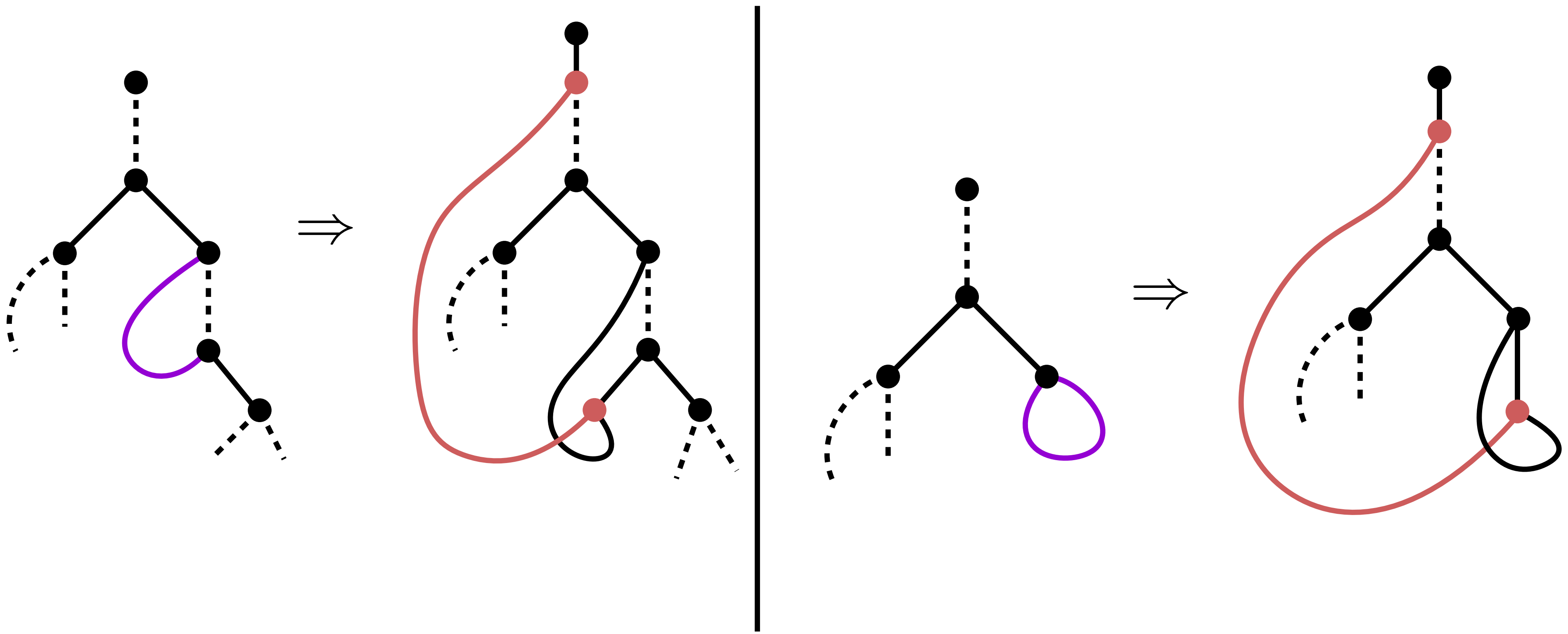
Cuts creating a p_1-pattern:



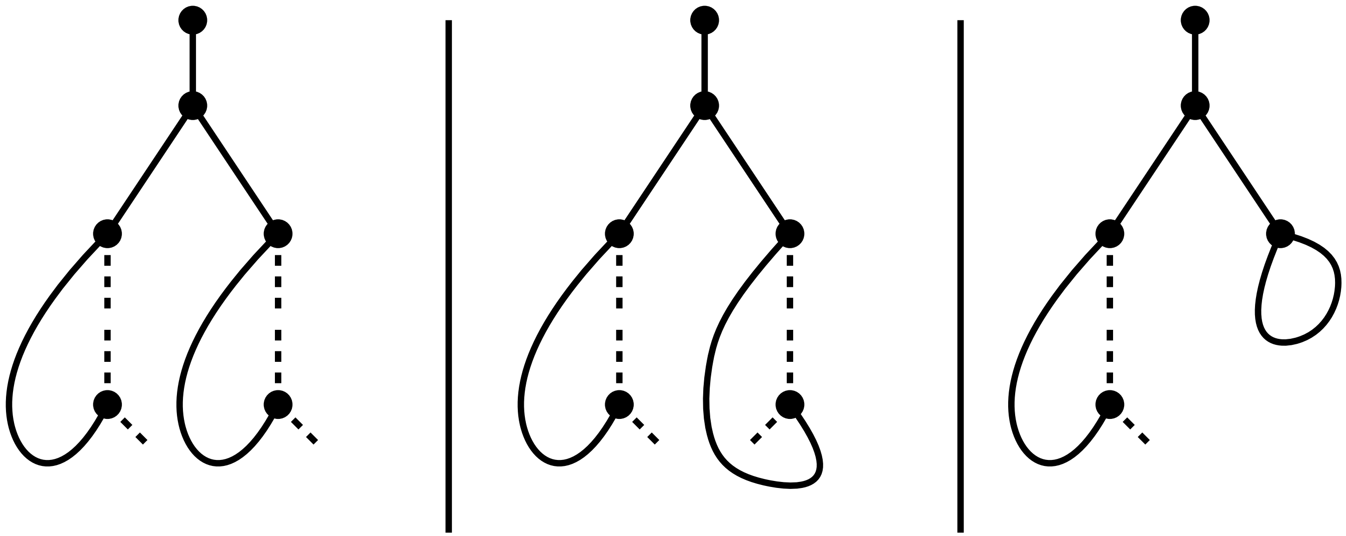
Thus we also need to keep track of:

$$C_1[\lambda x.C_2[(t_1\ x)])(\lambda y.t_2)] \qquad C_1[(\lambda x.x)(\lambda y.t_2)]$$

Enumerating p_1-patterns

- Tracking the creation/destruction of patterns during the recursive decomposition:

Applications creating p_1 and auxilliary patterns:



Thus, for an app. of the form $(l_1\ \lambda y.t_1)$ we need to consider how l_1 was formed.

Enumerating $\mathfrak{p}_1$-patterns

- Thus we have the following equations:

$$S = \Lambda + A$$

$$\Lambda = z^2 + 2z^4 S_z + (v - u + 4(1-u))z^3 S_u + (u - v + 4(1-v))z^3 S_v$$

$$A = zS^2 + (u-1)z(z^4 S_z + (v - u + 2(1-u))z^3 S_u + 2(1-v)z^3 S_v) \cdot \Lambda$$
$$+(v-1)z(z^2 + z^4 S_z + (u - v + 2(1-v))z^3 S_u + 2(1-u)z^3 S_u) \cdot \Lambda$$

- Extracting the mean:

$$\partial_u S|_{u=1,v=1}$$
$$= \left(2zS\partial_u S + 2z^4 \partial_{z,u} S + z^7 \partial_z S + 2z^9 (\partial_z S)^2 - 5z^3 \partial_u S + z^3 \partial_v S\right)|_{u=1,v=1}$$

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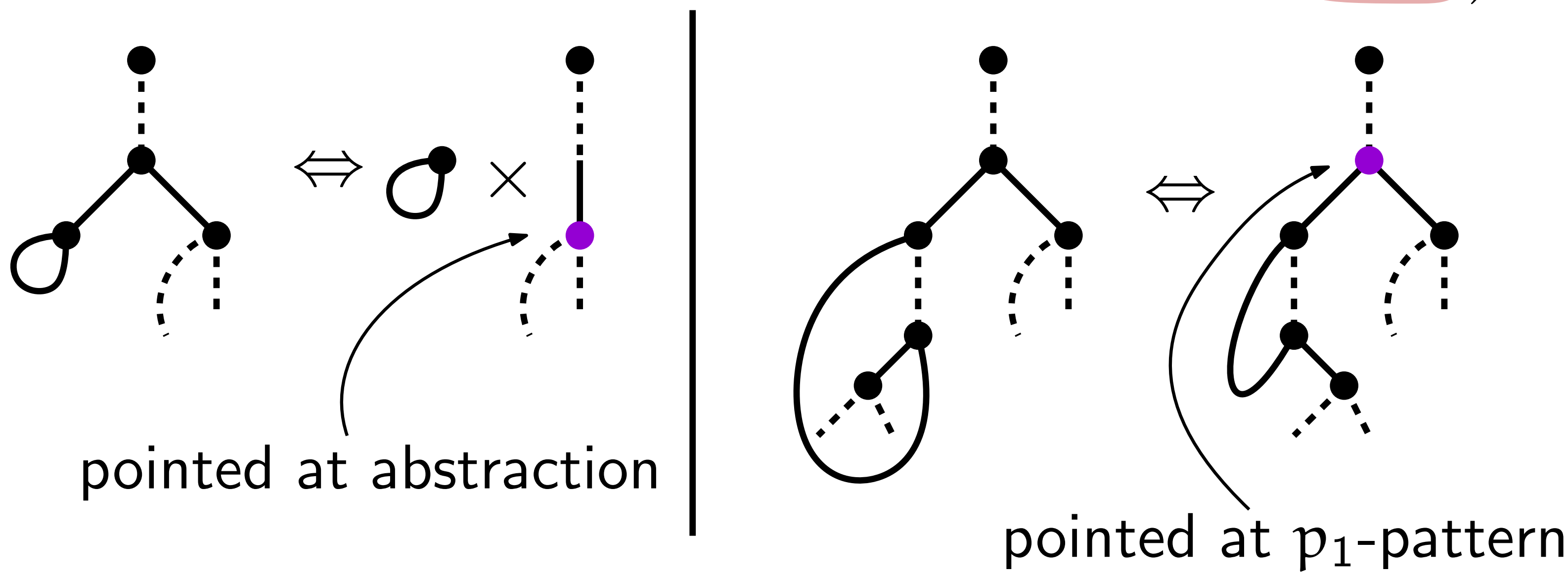
- Extracting the mean:

$\partial_u S|_{u=1,v=1}$ bijection: $\partial_v \leftrightarrow \partial_u$

$$= \left(2zS\partial_u S + 2z^4 \partial_{z,u} S + z^7 \partial_z S + 2z^9(\partial_z S)^2 - 5z^3 \partial_u S + z^3 \partial_v S\right)|_{u=1,v=1}$$



Enumerating $\mathfrak{p}_1$-patterns

- Finally we obtain a mean number of occurences:

$$\mathbb{E}[\#\ \mathfrak{p}_1 \text{ patterns}] \sim \frac{1}{6}$$

Enumerating $\mathfrak{p}_1$-patterns and $\mathfrak{p}_2$-patterns

- Finally we obtain a mean number of occurences:

$$\mathbb{E}[\# \ \mathfrak{p}_1 \text{ patterns}] \sim \frac{1}{6}$$

- Analogously, we have a mean number of occurences for $\mathfrak{p}_2$:

$$\mathbb{E}[\# \ \mathfrak{p}_2 \text{ patterns}] \sim \frac{1}{48}$$

Both are asymptotically constant in expectation!

Enumerating p_3-patterns

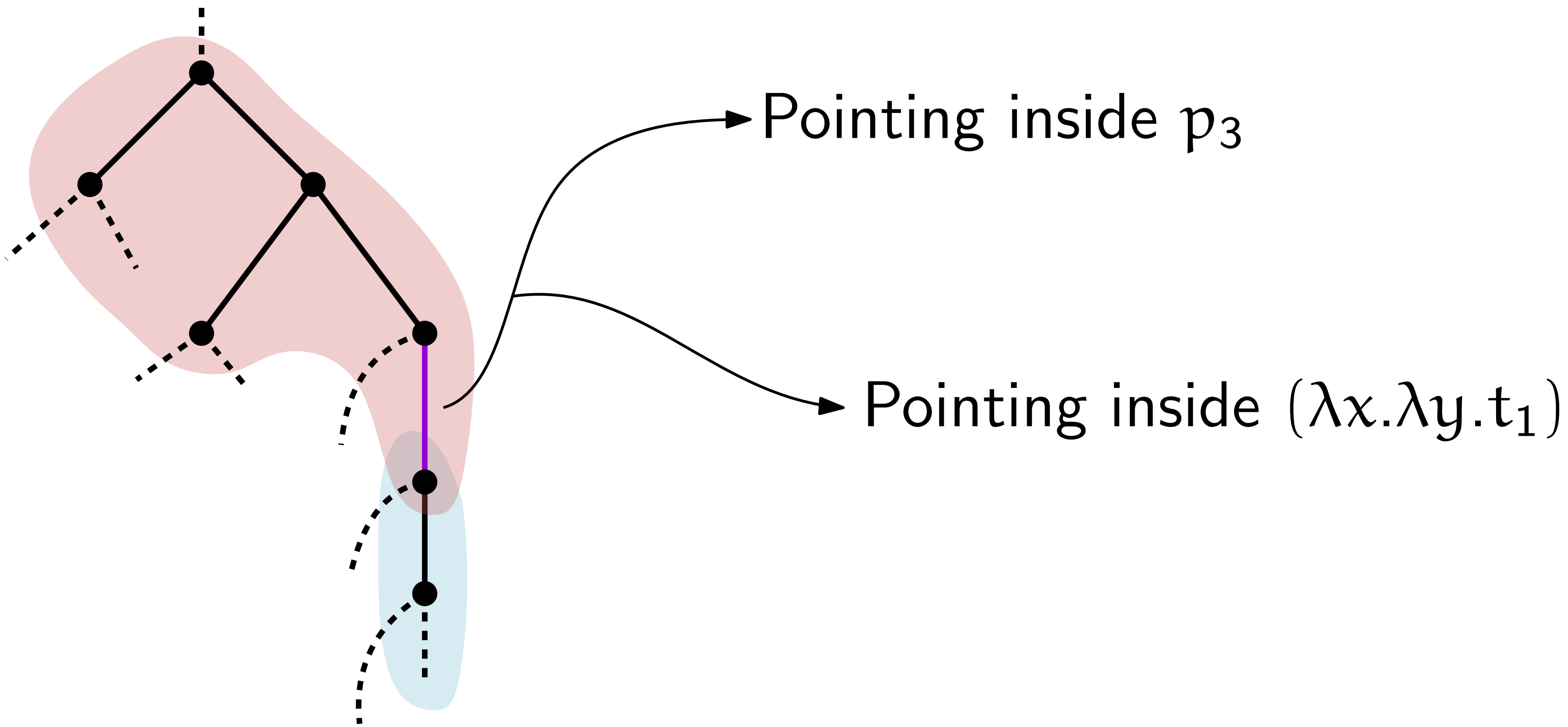
- As before, we'll also need to enumerate auxilliary patterns:

$(\lambda x.\lambda y.t_1)$

$(\lambda x.\lambda y.t_1)\ t_2$

$(\lambda x.\lambda y.t_1)\ t_2\ t_3 \quad (p_3)$

- However we run into a problem:



Enumerating $\mathfrak{p}_3$-patterns

- Generatingfunctionology fails, we revert to more elementary methods:

$$\mathbb{E}(V_n) = \mathbb{E}(V_n|\Lambda_n) \cdot \frac{|\Lambda_n|}{|L_n|} + \mathbb{E}(V_n|A_n) \cdot \frac{|A_n|}{|L_n|}$$

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Magic: linear over *families* of all possible abstractions created via cuts from a fixed term!

$$\overline{X}_n = (2n - 12)\overline{X}_{n-3}2\overline{Y}_{n-3}$$
$$\overline{Y}_n = (2n - 6)Y_{n-3} - 6Y_{n-3}$$
$$\overline{Z}_n = 2(n - 4)(Z + \mathbf{1}_{\Lambda_n})$$

where: X_n counts # of p_1 patt. over terms of size n

Y_n is the same for the pattern $(\lambda x.\lambda y.t_1)\ t_2$, and

Z is the same for the pattern $(\lambda x.\lambda y.t_1)$

The $\overline{V}$ for $V \in \{X_n, Y_n, Z_n\}$ are cummulatives over families of abstractions

The lower bound

Theorem

Let W_n be the random variable given by number of steps required to normalise a linear term of size $n \in 3\mathbb{N} + 2$. Then

$$\mathbb{E}(W_n) \geqslant \tfrac{11n}{240}, \text{ for } n \text{ large enough}$$

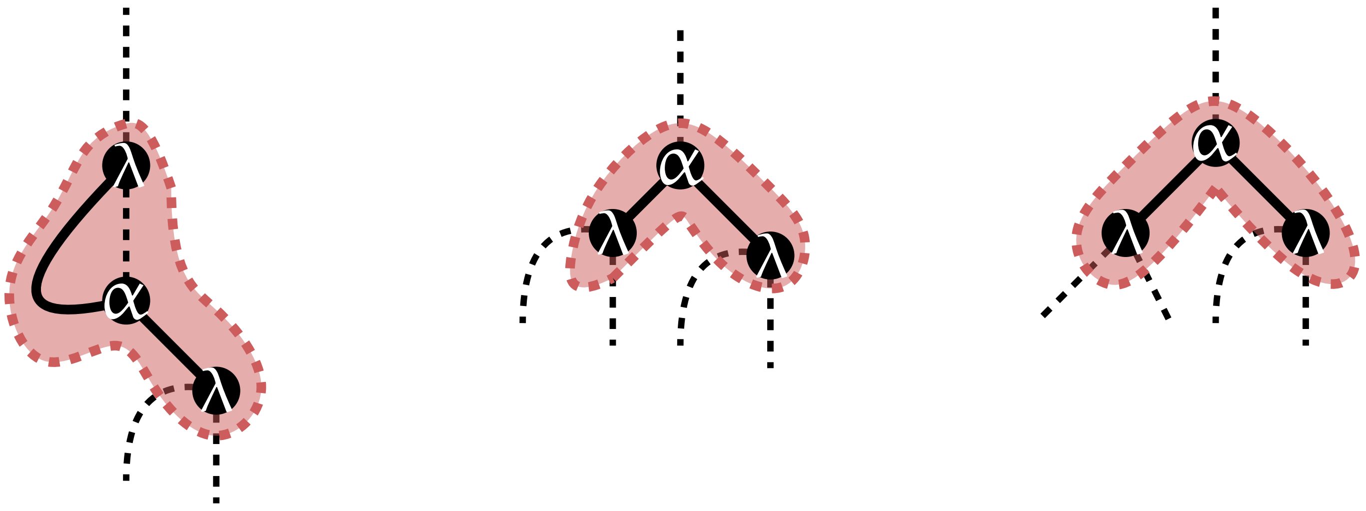
During an open problem session of CLA 2020, Noam Zeilberger conjectured:

$$\mathbb{E}(W_n) \sim \frac{n}{21}$$

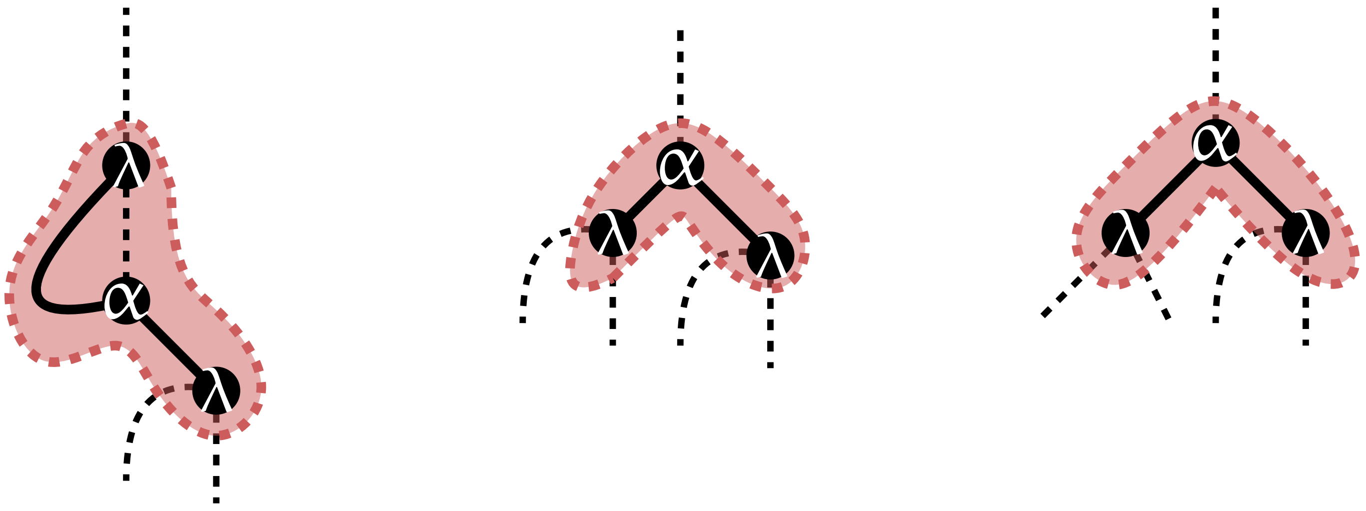
We got pretty close:

$$\frac{11}{240} - \frac{1}{21} = 0.001\ldots$$

Counting redices by type of argument



Counting redices by type of argument



Theorem

$\mathbb{E}(\#(\lambda x.t)\ y) \sim \frac{n}{30}$

$\mathbb{E}(\#(\lambda x.t)\ (\lambda y.t')) \sim \frac{1}{20}$

$\mathbb{E}(\#(a\ b)\ (\lambda y.t')) \sim \frac{n}{120}$

Open problems

- Classify patterns according to their expected number of occurences: constant or linear in n.

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Thank you!

Bibliography

[BGGJ13] Bodini, O., Gardy, D., Gittenberger, B., & Jacquot, A. (2013).
Enumeration of Generalized BCI Lambda-terms.
The Electronic Journal of Combinatorics, P30-P30.

[Z16] Zeilberger, N. (2016).
Linear lambda terms as invariants of rooted trivalent maps.
Journal of functional programming, 26.

[AB00] Arques, D., & Béraud, J. F. (2000).
Rooted maps on orientable surfaces, Riccati's equation and continued fractions
Discrete mathematics, 215(1-3), 1-12.

[BFSS01] Banderier, C., Flajolet, P., Schaeffer, G., & Soria, M. (2001).
Random maps, coalescing saddles, singularity analysis, and Airy phenomena.
Random Structures & Algorithms, 19(3-4), 194-246.

Bibliography

[BR86] Bender, E. A., & Richmond, L. B. (1986).
A survey of the asymptotic behaviour of maps.
Journal of Combinatorial Theory, Series B, 40(3), 297-329.

[BGLZ16] Bendkowski, M., Grygiel, K., Lescanne, P., & Zaionc, M. (2016).
A natural counting of lambda terms.
In International Conference on Current Trends in Theory and Practice of Informatics (pp. 183-194). Springer, Berlin, Heidelberg.

[BBD19] Bendkowski, M., Bodini, O., & Dovgal, S. (2019).
Statistical Properties of Lambda Terms.
The Electronic Journal of Combinatorics, P4-1.

[BCDH18] Bodini, O., Courtiel, J., Dovgal, S., & Hwang, H. K. (2018, June).
Asymptotic distribution of parameters in random maps.
In 29th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms (Vol. 110, pp. 13-1)

Bibliography

[B75] Bender, E. A. (1975).
An asymptotic expansion for the coefficients of some formal power series.
Journal of the London Mathematical Society, 2(3), 451-458.

[FS93] Flajolet, P., & Soria, M. (1993).
General combinatorial schemas: Gaussian limit distributions and exponential tails.
Discrete Mathematics, 114(1-3), 159-180.

[B18] Borinsky, M. (2018).
Generating Asymptotics for Factorially Divergent Sequences.
The Electronic Journal of Combinatorics, P4-1.

[BKW21] Banderier, C., Kuba, M., & Wallner, M. (2021).
Analytic Combinatorics of Composition schemes and phase transitions
mixed Poisson distributions.
arXiv preprint arXiv:2103.03751.

Bibliography

[BGJ13] Bodini, O., Gardy, D., & Jacquot, A. (2013).
Asymptotics and random sampling for BCI and BCK lambda terms
Theoretical Computer Science, 502, 227-238.

[M04] Mairson, H. G. (2004).
Linear lambda calculus and PTIME-completeness
Journal of Functional Programming, 14(6), 623-633.

[DGKRTZ13] Zaionc, M., Theyssier, G., Raffalli, C., Kozic, J.,
J., Grygiel, K., & David, R. (2013)
Asymptotically almost all λ-terms are strongly normalizing
Logical Methods in Computer Science, 9

[SAKT17] Sin'Ya, R., Asada, K., Kobayashi, N., & Tsukada, T. (2017)
Almost Every Simply Typed λ-Term Has a Long β-Reduction Sequence
In International Conference on Foundations of Software Science and
and Computation Structures (pp. 53-68). Springer, Berlin, Heidelberg.

Bibliography

[B19] Baptiste L. (2019).
A new family of bijections for planar maps
Journal of Combinatorial Theory, Series A.