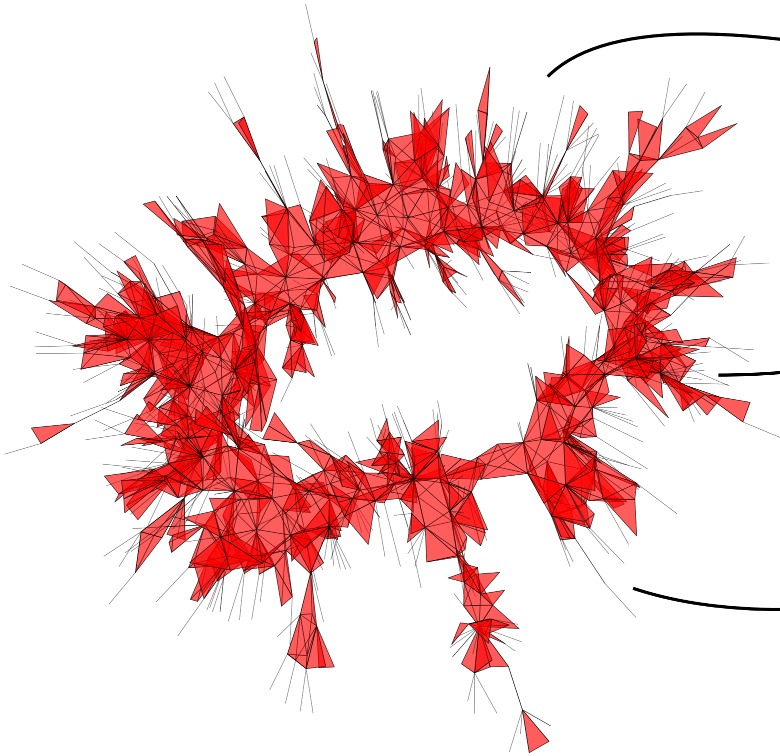
On some functional equations for maps



$$(T = uz + zT^2 + z\partial_u T)|_{u=0}$$

$$T = z^2 + zT^2 + 2z^4\partial_z T$$

$$(T = (v-1)z^2 + zT^2 + z\partial_v T)|_{v=1}$$

Alexandros Singh (Université Paris 8)

Based on joint work(s) with Olivier Bodini and Konstantinos Tsagkaris

Topical day: Elimination for Functional Equations

December 11, 2023

Outline

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- Presentation of maps

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- Some of our results on statistics/parameters of maps

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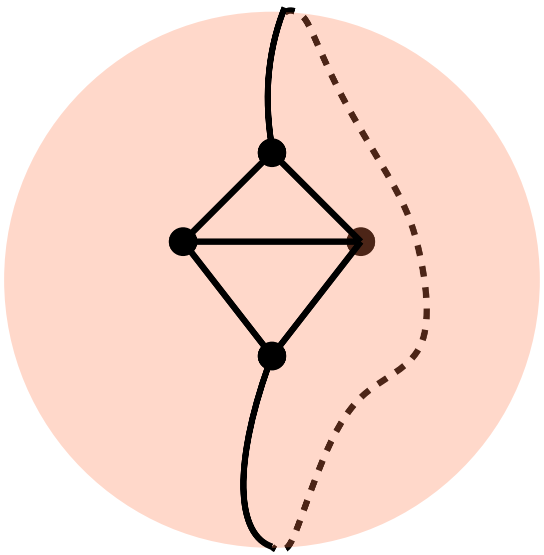
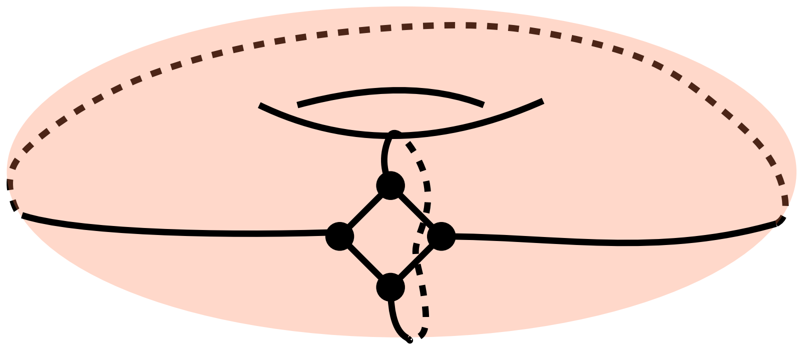
- “Guessing” and relating functional equations
- Questions for computer algebraists

What are maps?

Cellular embeddings of (multi)graphs on surfaces.

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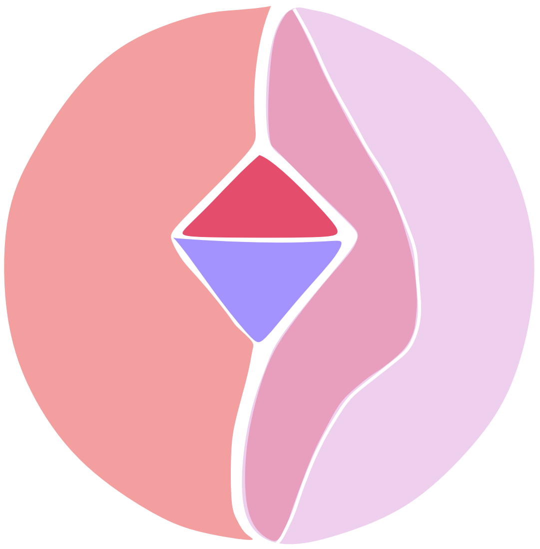

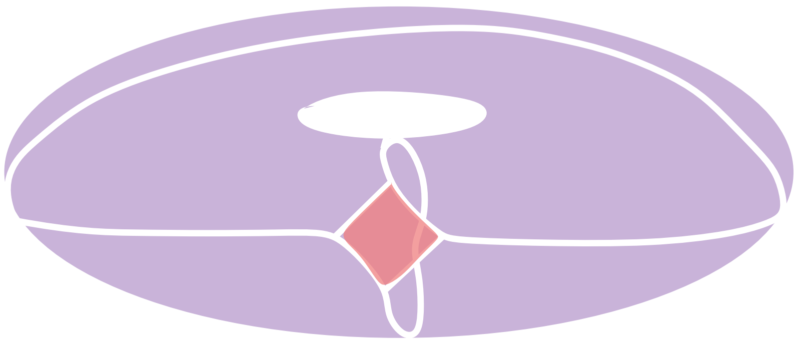

Cellular embeddings of (multi)graphs on surfaces.

↳ faces homeomorphic to open disks

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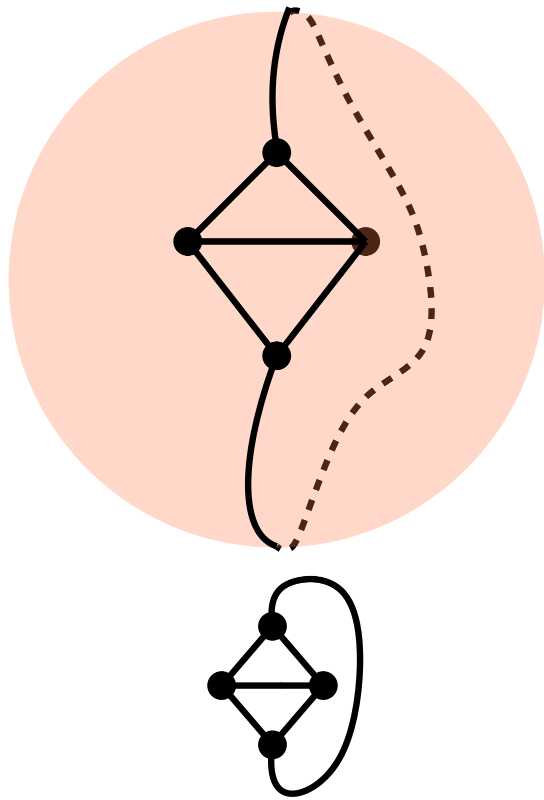

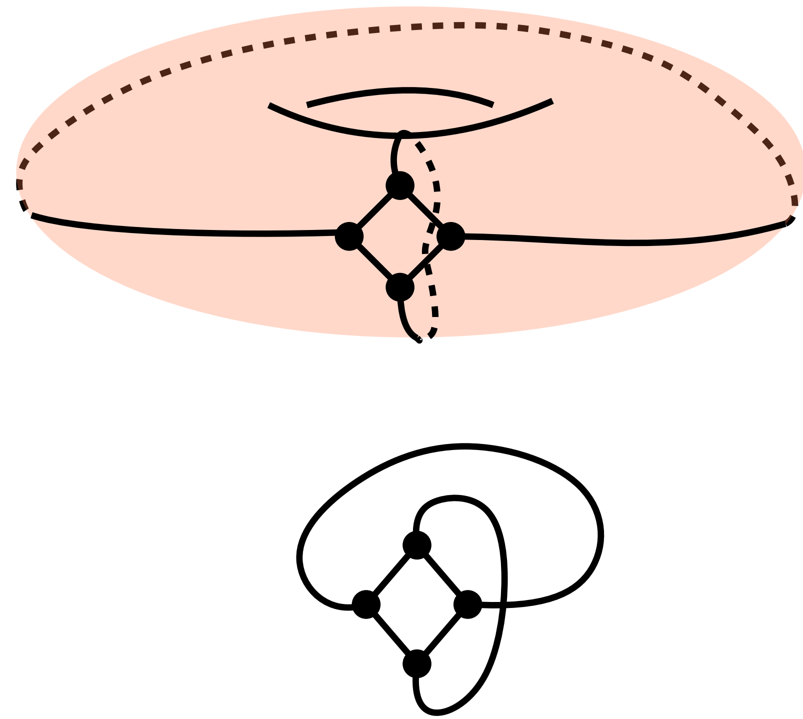

4CT...

- A central object in modern combinatorics, but not only that: probability, algebraic geometry, theoretical physics...

scaling limits... matrix integrals, Witten's conjecture, . . .

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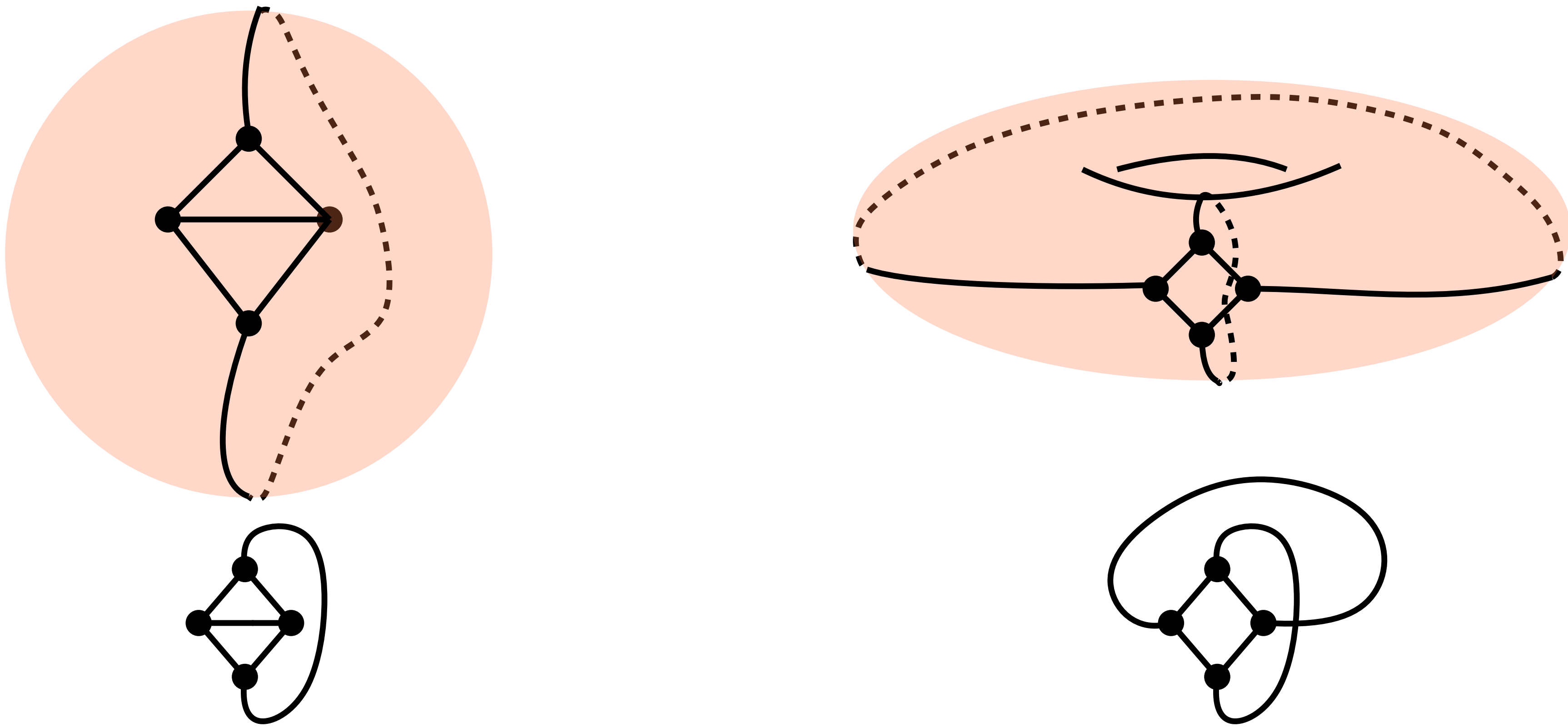
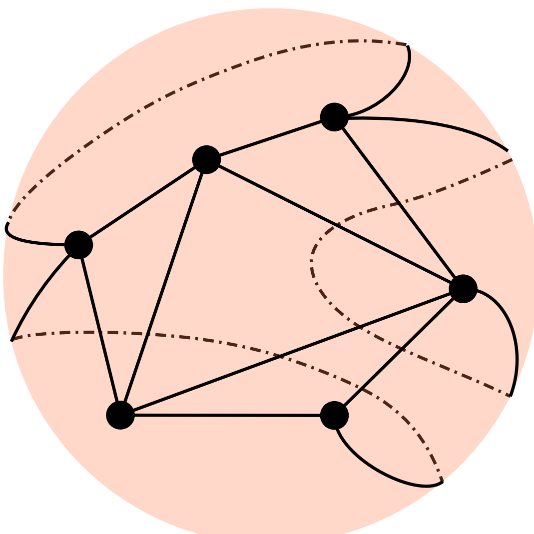


- A central object in modern combinatorics, but not only that: probability, algebraic geometry, theoretical physics...
- Their enumeration was pioneered by Tutte in the 60s, as part of his approach to the four colour theorem.

Triangulations and trivalent maps

Triangulations and trivalent maps

A much studied class: maps where all faces are of degree 3



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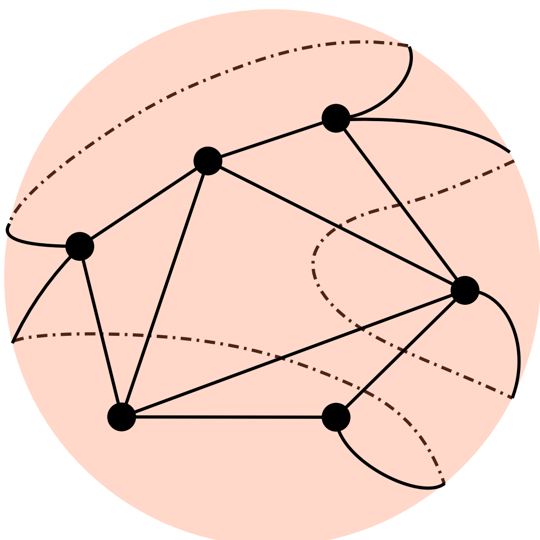

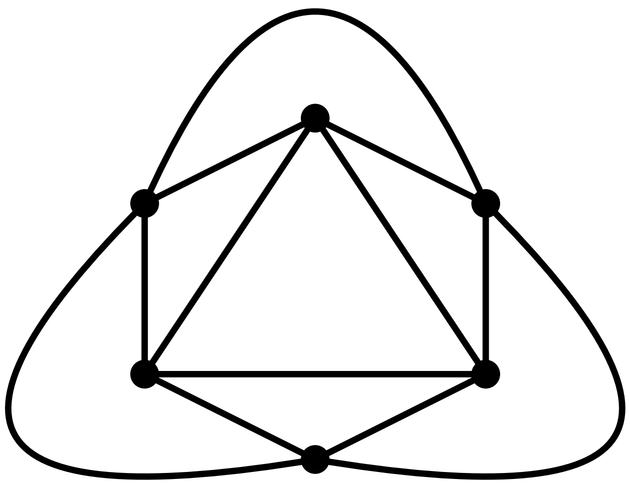

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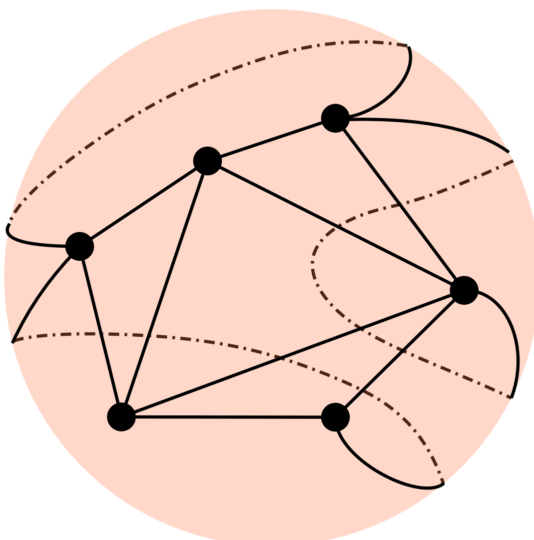

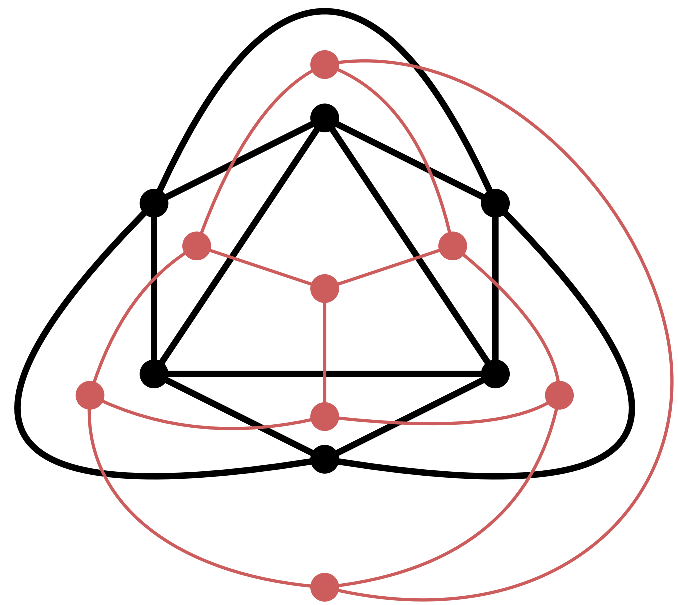

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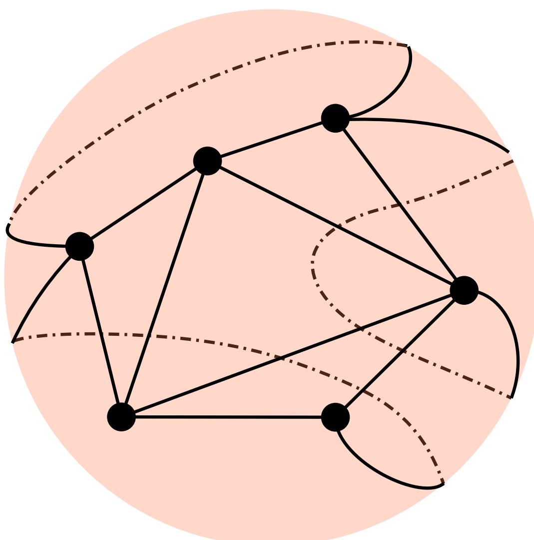
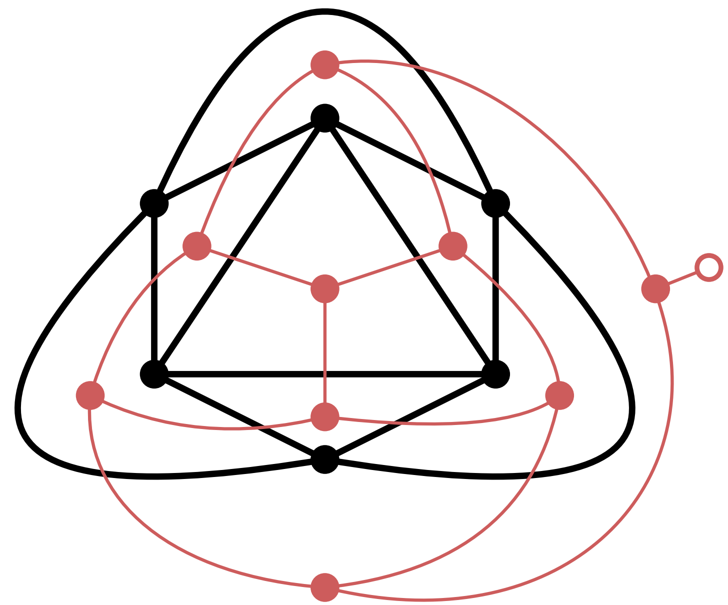
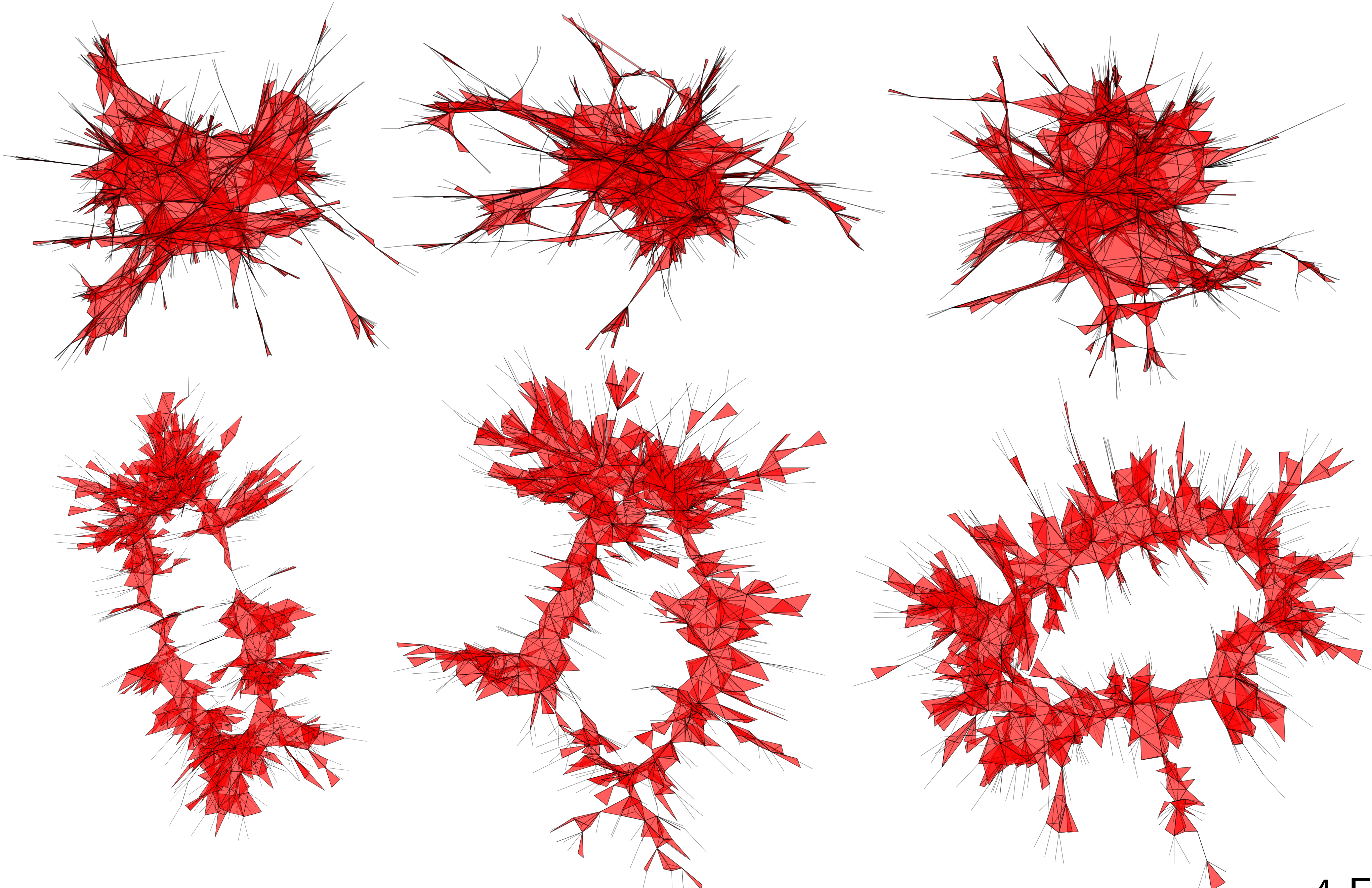




Rooting them makes it easier to count.

Triangulations and trivalent maps

Random triangulations of the sphere and torus with ≈ 3000 triangles:



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- QFT in zero dimensions [CLP78]

$$\mathcal{Z} = \int e^{-\left(\frac{\phi^2}{2} + \frac{z\phi^3}{3}\right) + J\phi} d\phi, \ \langle \phi \rangle_{J=0} =$$

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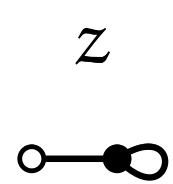

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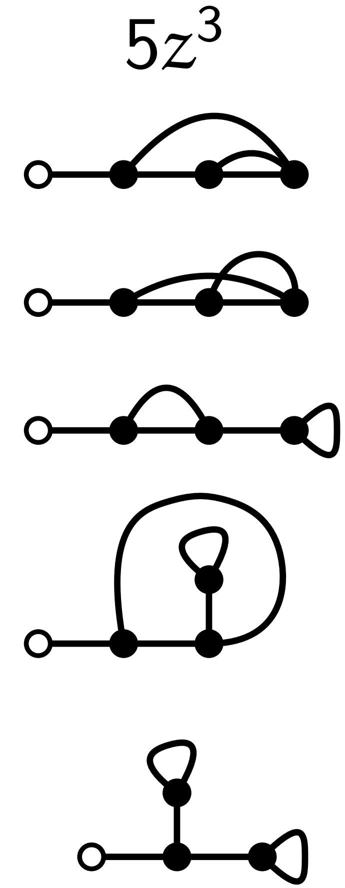

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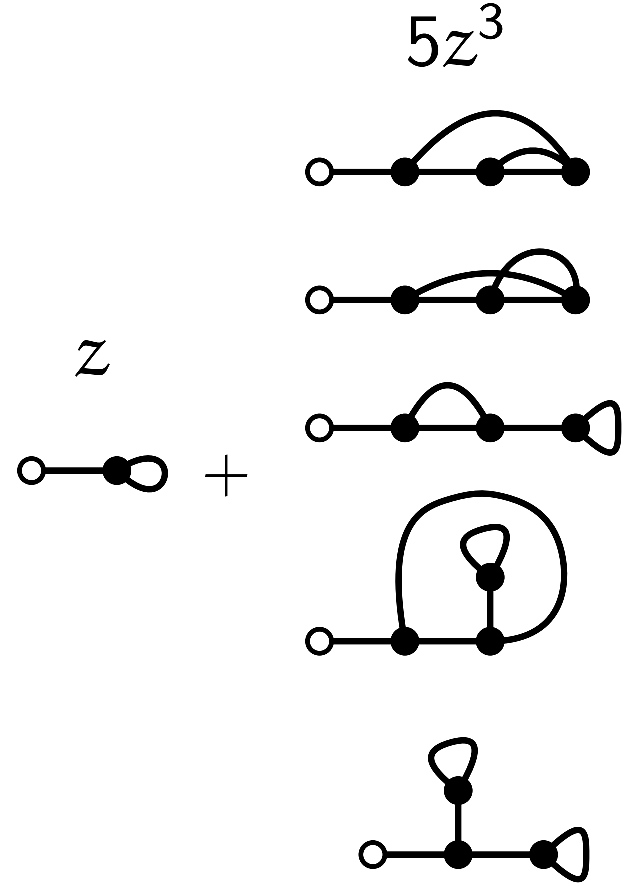

$+ \ldots$

(Do a matrix integral if you want maps sorted by genus!)

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$$(\lambda x.x)\ (\lambda y.(\lambda z.z\ y)\ (\lambda w.\lambda u.w\ u))$$

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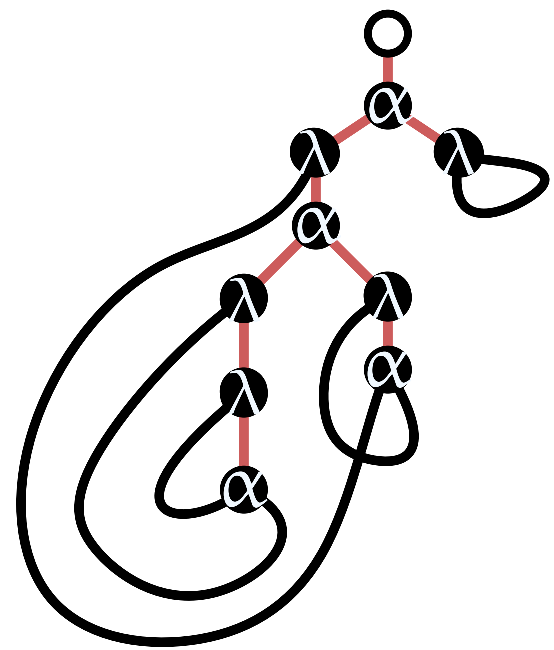

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Algebra:

- Combinatorics of subgroups of the modular group $\mathrm{PSL}(2;\mathbb{Z})$ [HMR16]

Combinatorial questions

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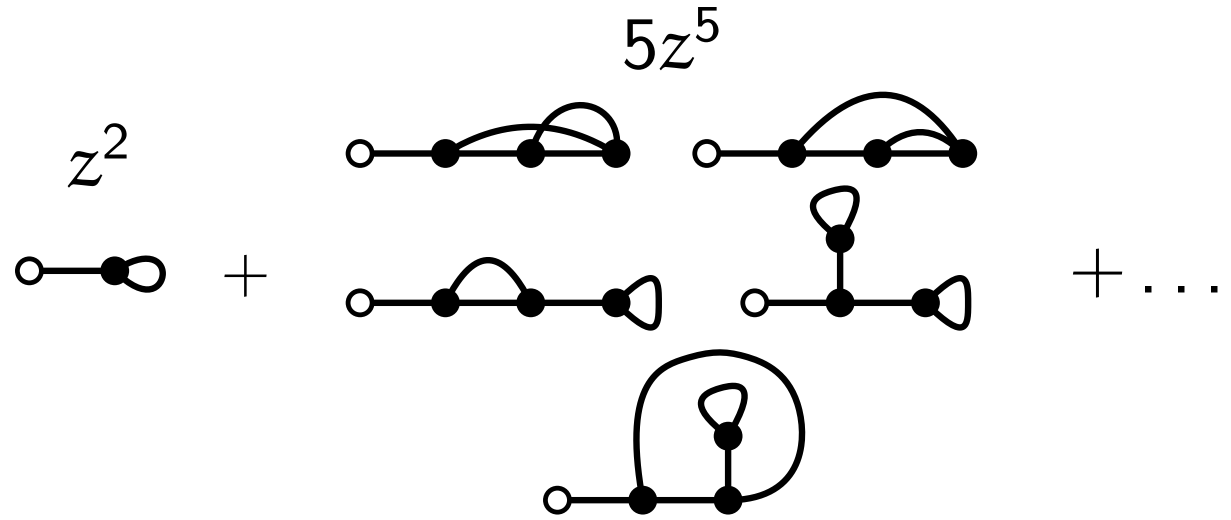
- Counting via generating functions

size = # edges

$$T(z) = \sum_{t \in T} z^{|t|} =$$



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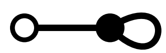
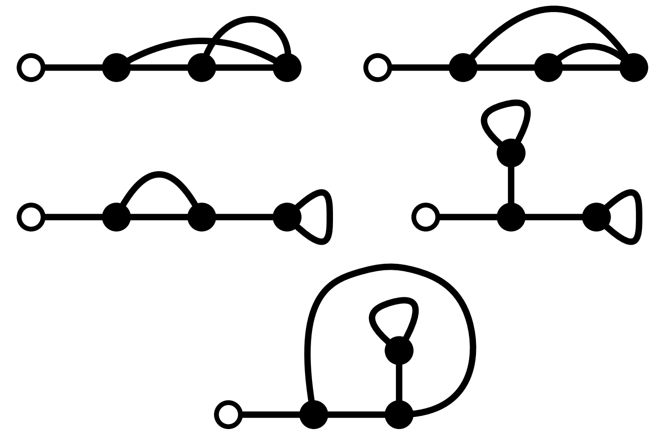
$5z^5$

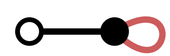
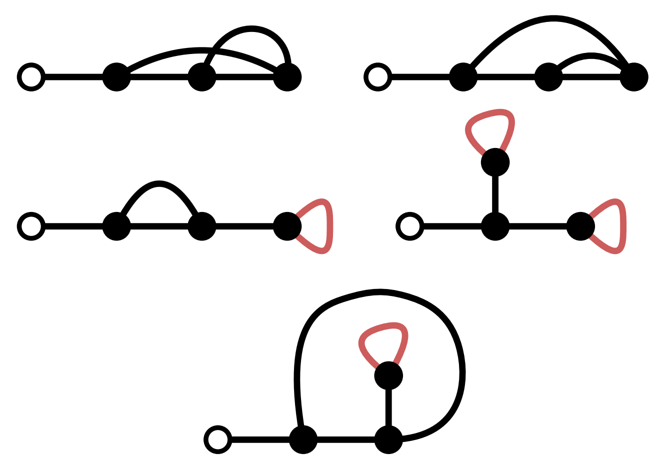
z^2

$T(z) = \sum_{t \in T} z^{|t|} =$  $+$  $+ \dots$

- "Advanced counting": combinatorial parameters, observables

$(v^2 + 2v + 2)z^5$

vz^2

$T(z, v) = \sum_{t \in T} z^{|t|} v^{\#\text{loops}} =$  $+$  $+ \dots$

Some results [BSZ21,S22]

● =w. Bodini, Zeilberger ● = ● + Gittenberger, Wallner

Parameters on maps and terms of arbitrary genus (number of):

- ● Loops in trivalent maps and identity-subterms in closed linear terms

  Limit law: $\mathrm{Poisson}(1)$

- ● Bridges in trivalent maps and closed subterms in closed linear terms

  Limit law: $\mathrm{Poisson}(1)$

- ● Vertices of degree 1 in (1,3)-valent maps and free variables in open linear terms

  Limit law: $\mathcal{N}((2n)^{1/3}, (2n)^{1/3})$

- ● Patterns in trivalent maps and redices in closed linear terms

  Asymptotic mean and variance: $\frac{n}{24}$

- ● Steps to reach normal form for closed linear terms

  Asymptotic mean bound below by: $\frac{11n}{240}$

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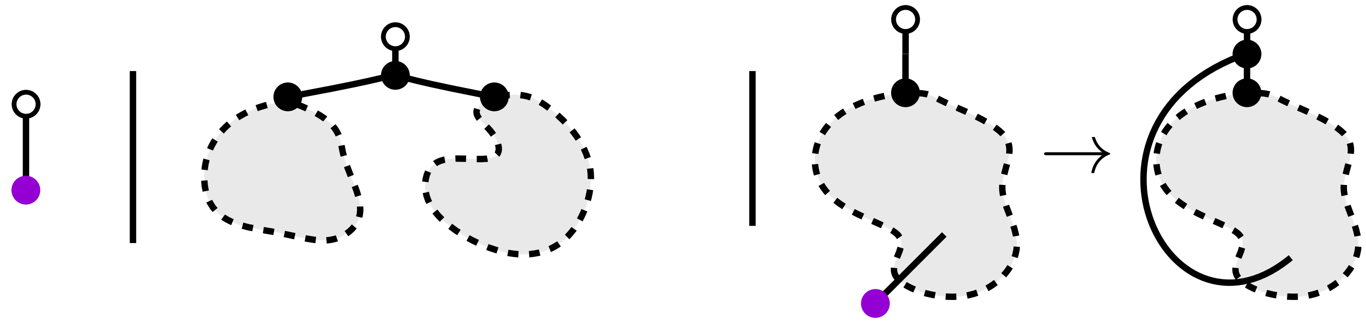
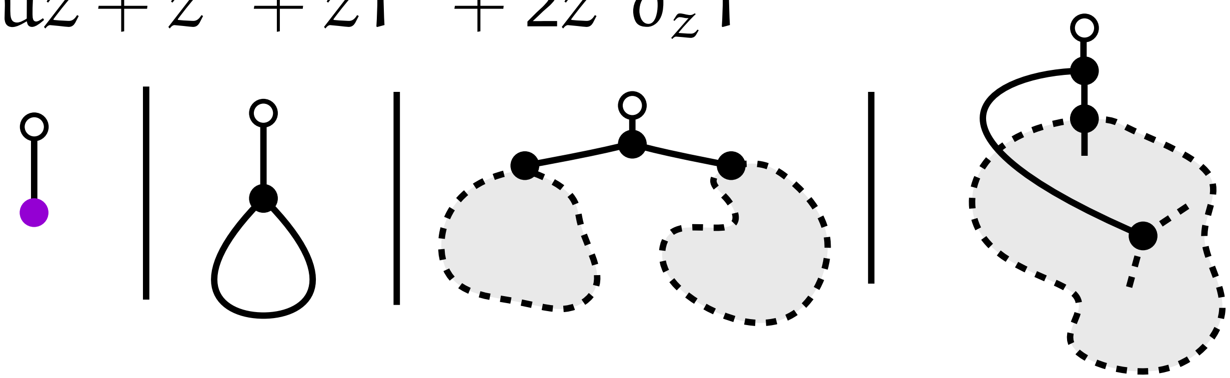
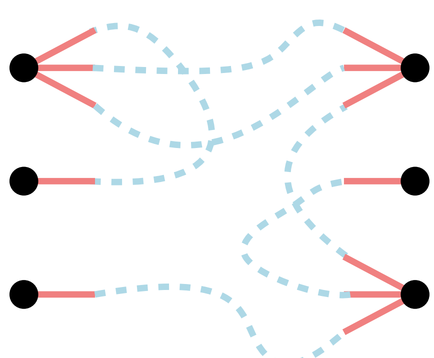


- $T = uz + z^2 + zT^2 + 2z^4\partial_z T$



- $T = uz^2 + z^4 + z^5 \frac{\partial}{\partial z} \left(\ln \left(\exp(z^2/2) \odot \exp(z^3/3 + uz)\right)\right)$



Our strategy:

1) Track evolution of parameters through decompositions of maps/λ-terms

different decompositions $\rightsquigarrow$ differential equations, Hadamard products, . . .

generating functions divergent away from 0

2) Develop tools for rapidly growing coefficients, based on:

- Moment pumping
- Bender's theorem for compositions $F(z, G(z))$ [B75]
- Coefficient asymptotics of Cauchy products

$$[z^n](A(z) \cdot B(z)) \sim a_n b_0 + a_0 b_n + O(a_{n-1} + b_{n-1})$$

for A, B, G divergent and F analytic

Decomposing rooted open trivalent maps

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edges

$$T(z,u) = uz$$

unary vertices

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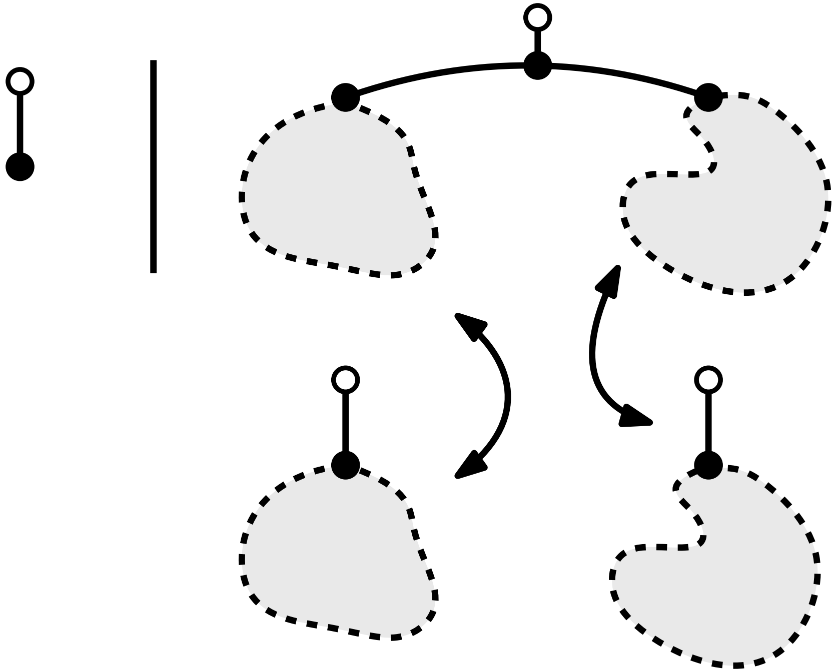
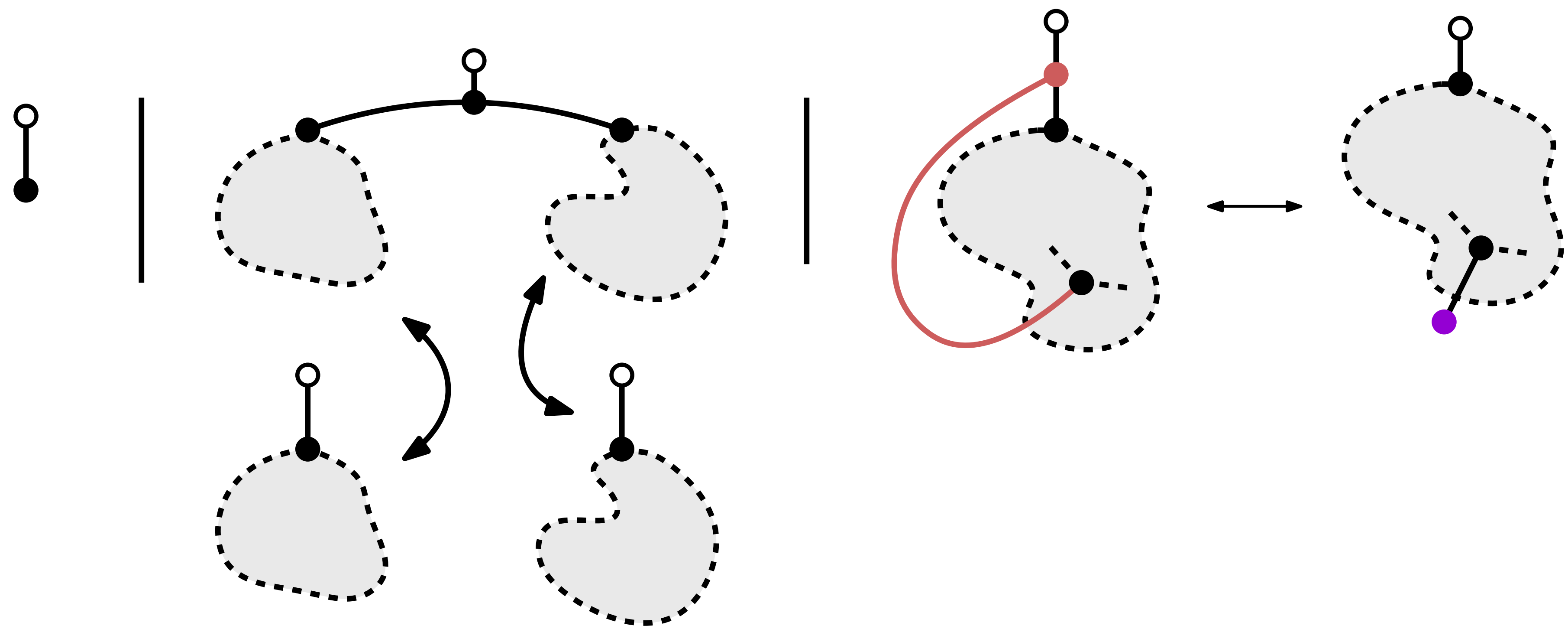


edges

$$\mathsf{T}(z, u) = uz + z\mathsf{T}(z, u)^2$$

unary vertices

Decomposing rooted open trivalent maps



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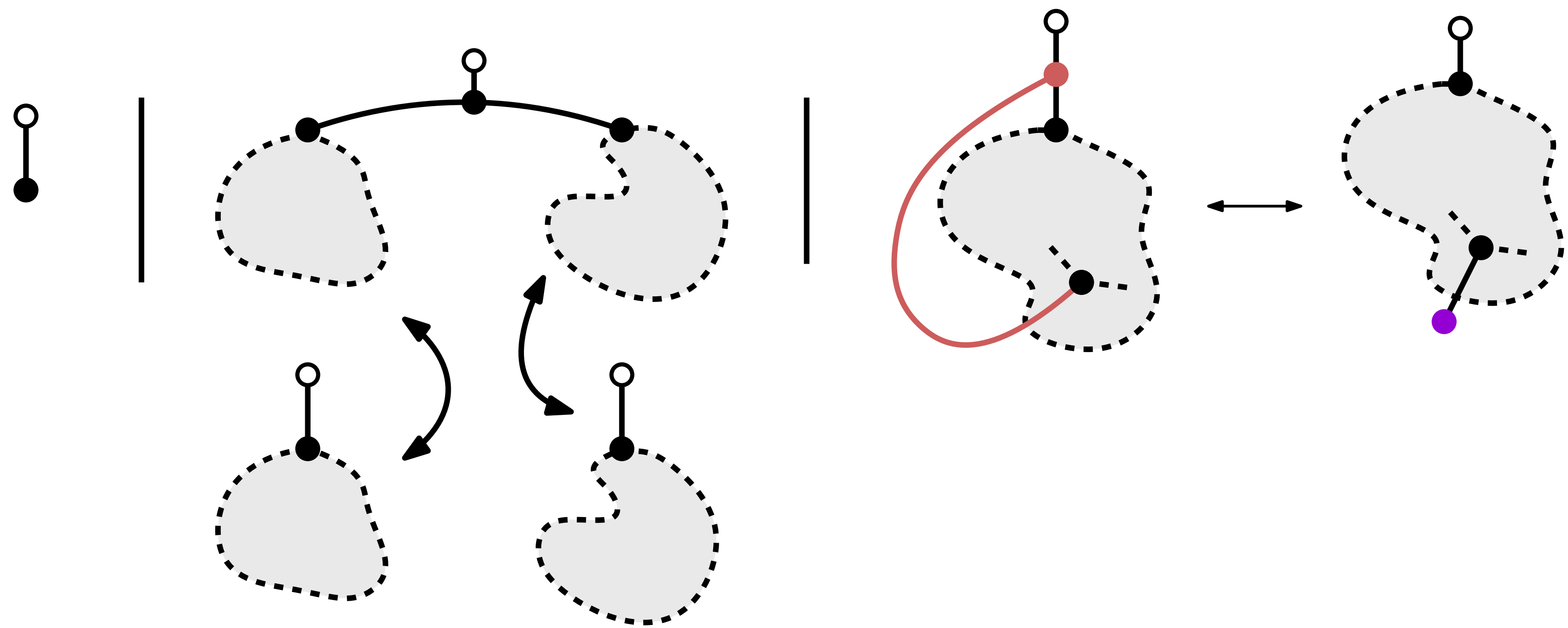

See also: Schwinger-Dyson eq. of

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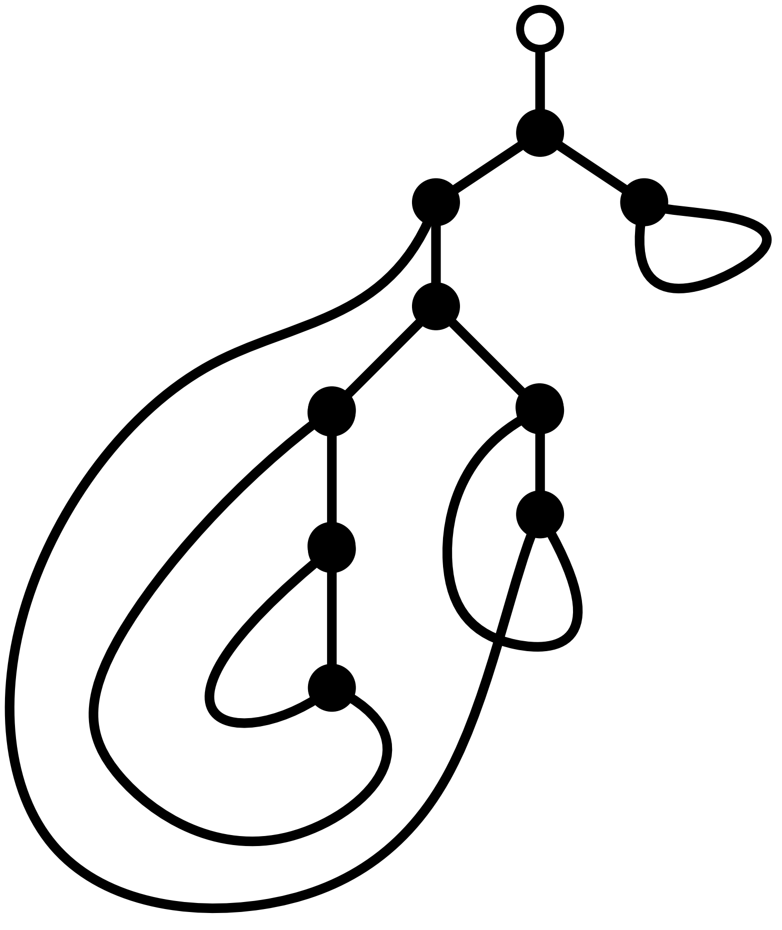

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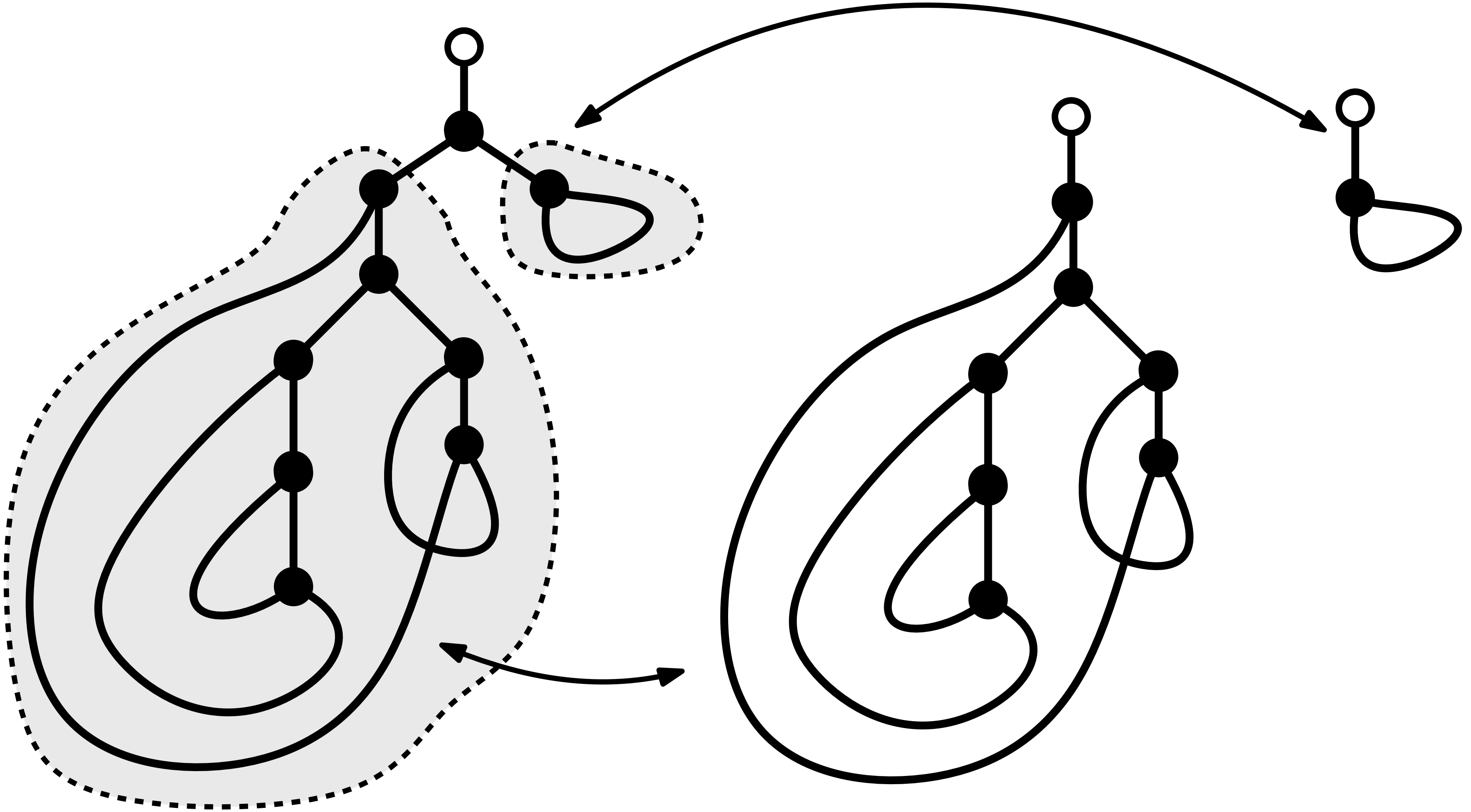

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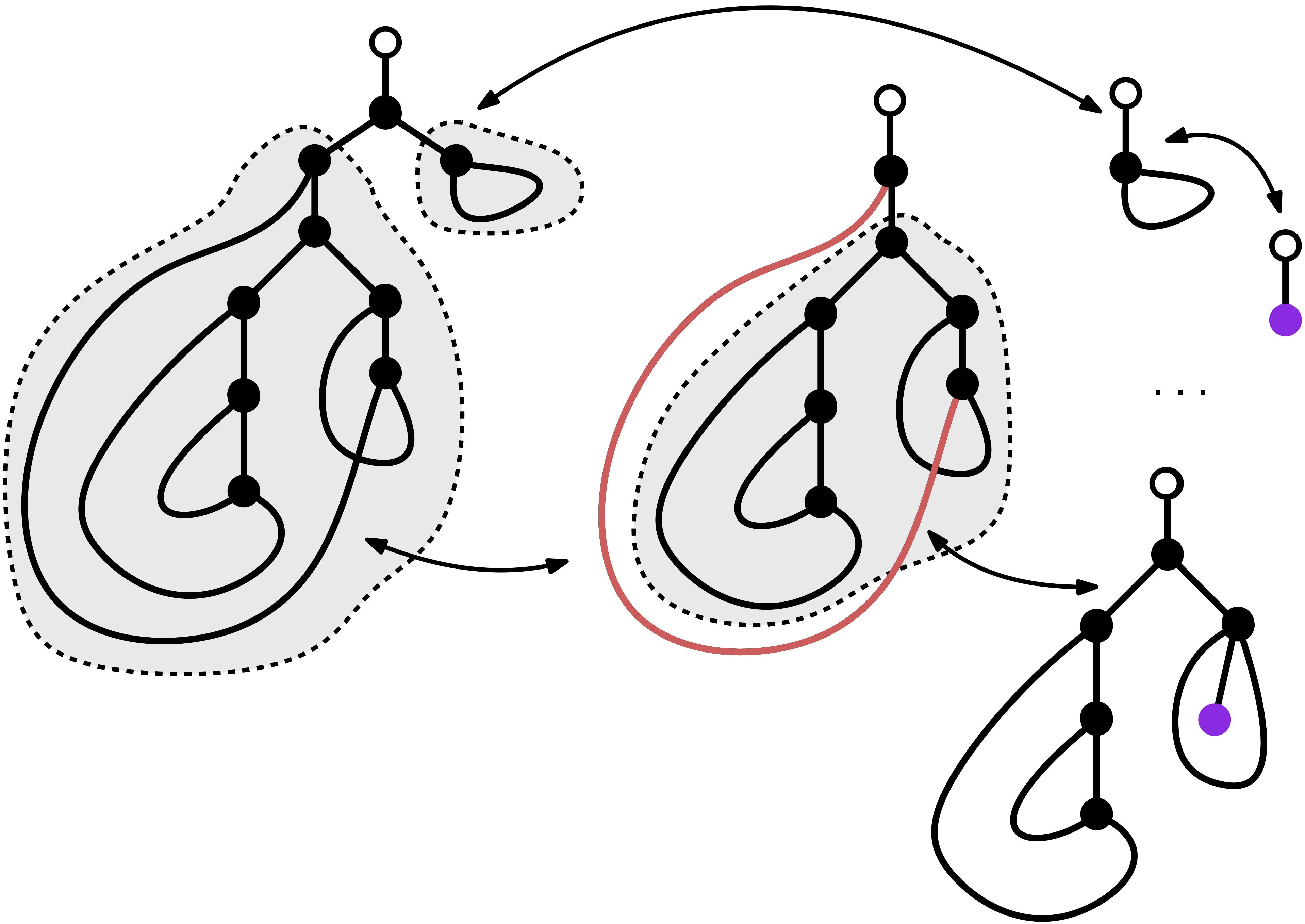

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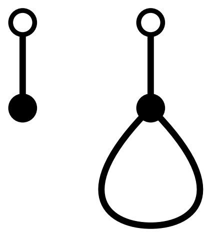
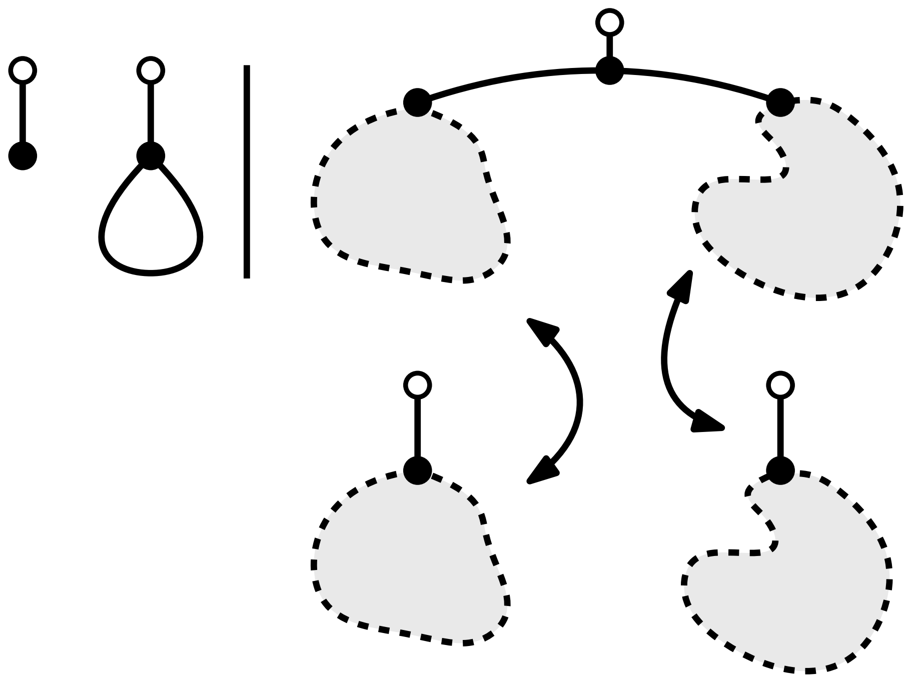


$$T(z, u) = uz + z^2$$

Decomposing rooted open trivalent maps, again



$$\mathsf{T}(z, u) = uz + z^2 + zT(z)^2$$

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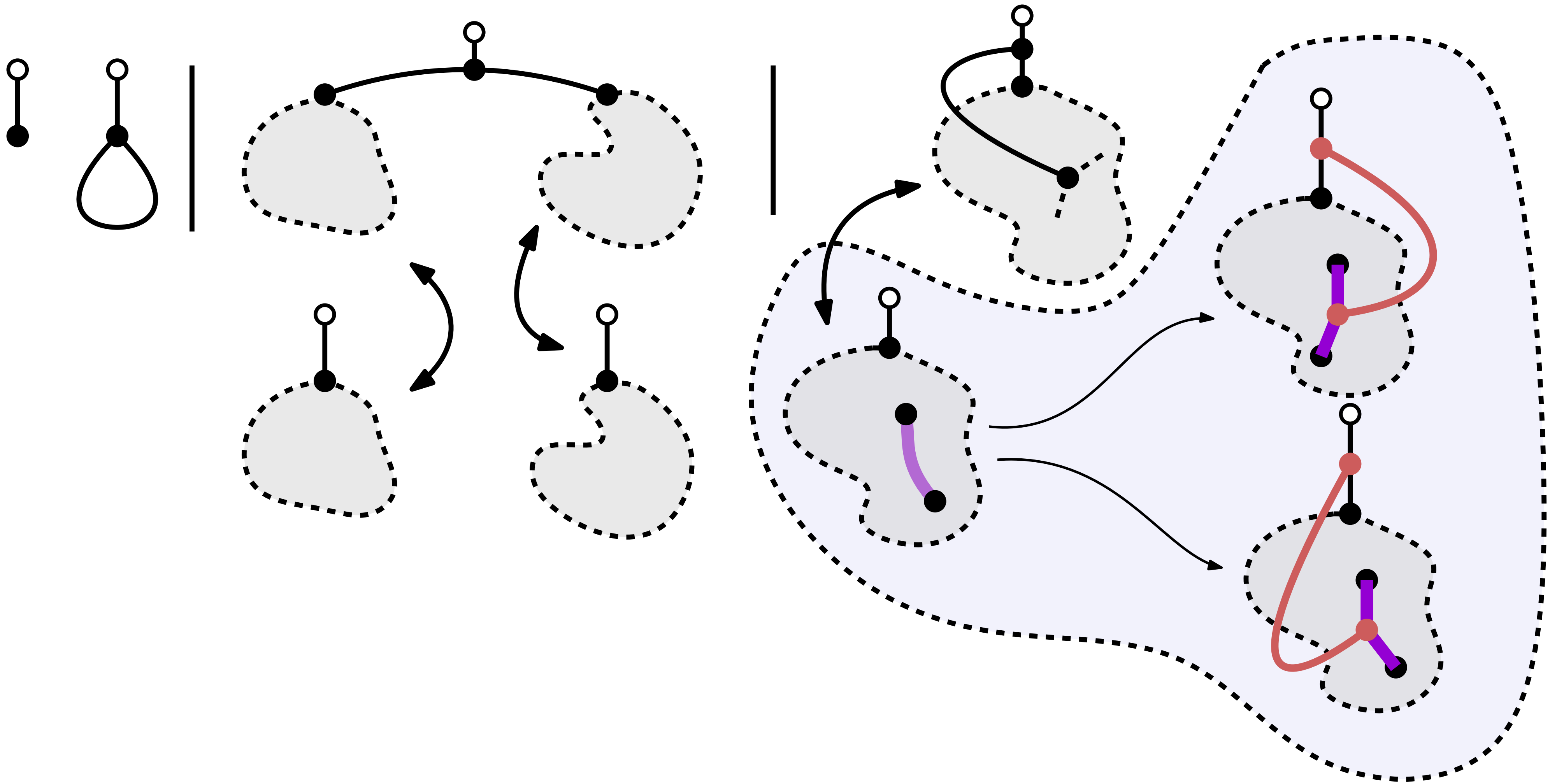

$$T(z,u) = uz + z^2 + zT(z)^2 + 2z^4\partial_z T(z,u)$$

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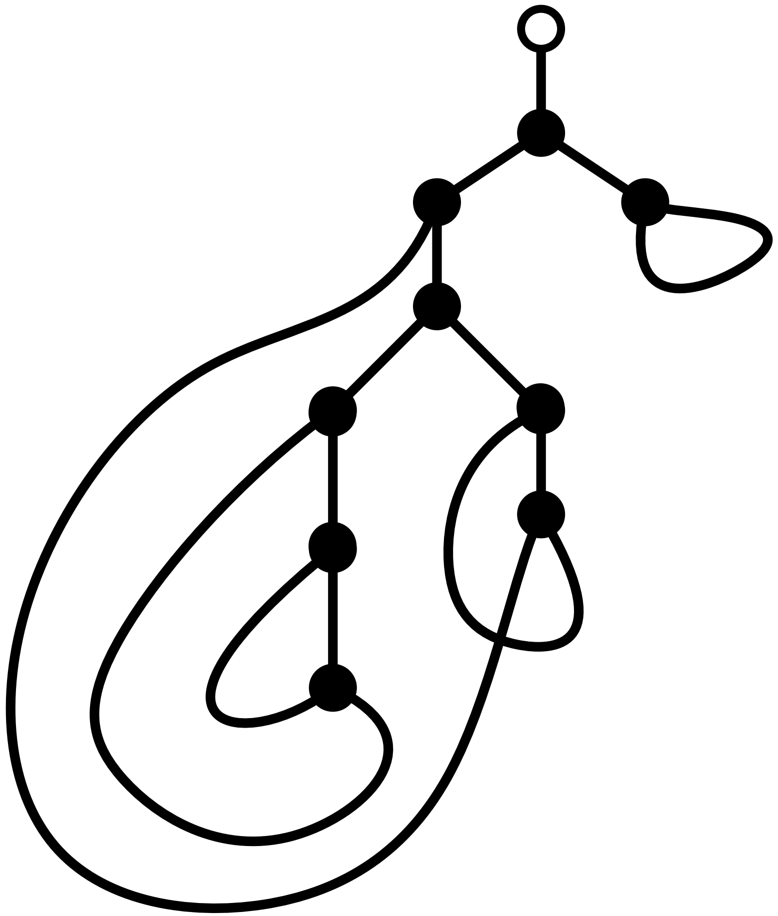

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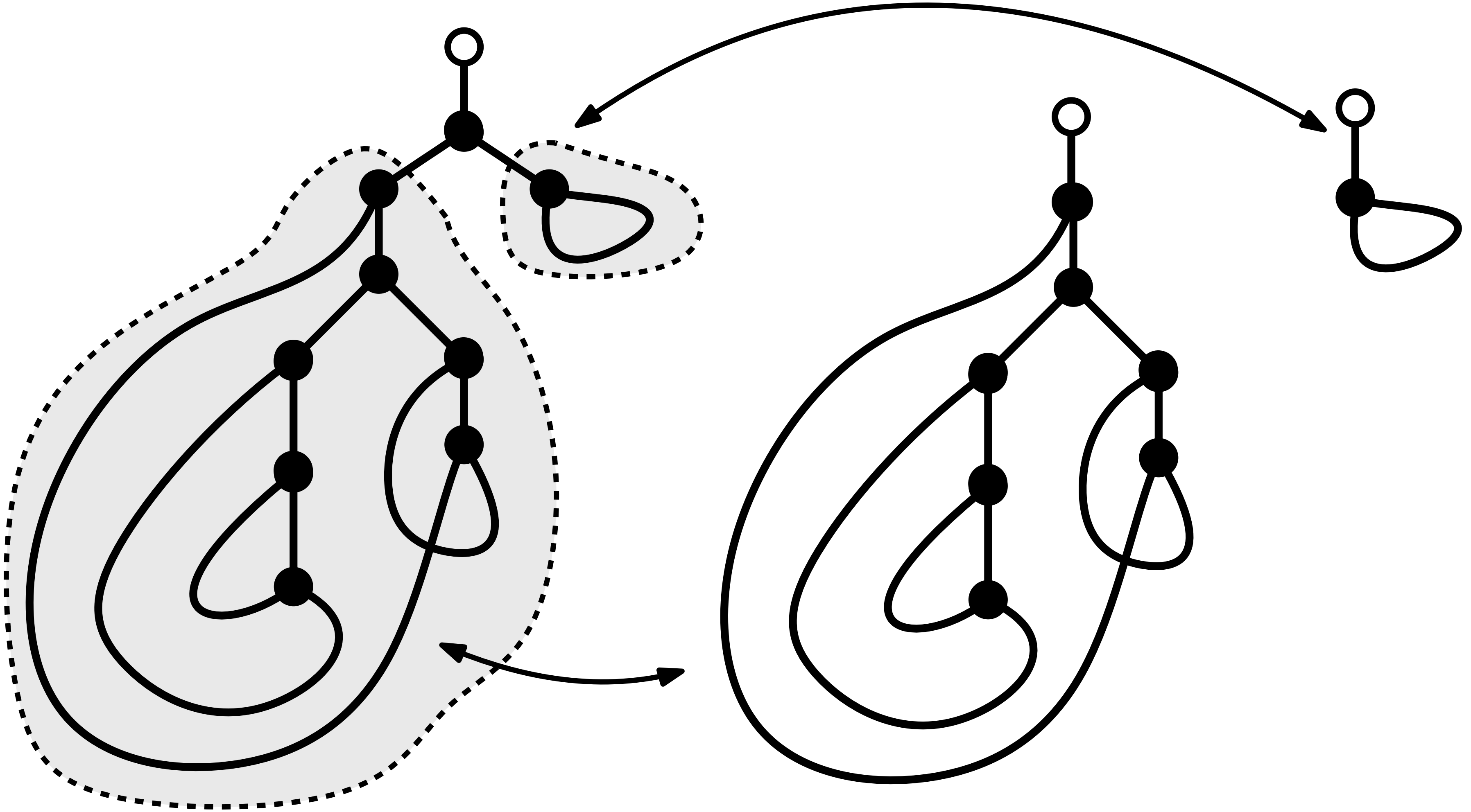

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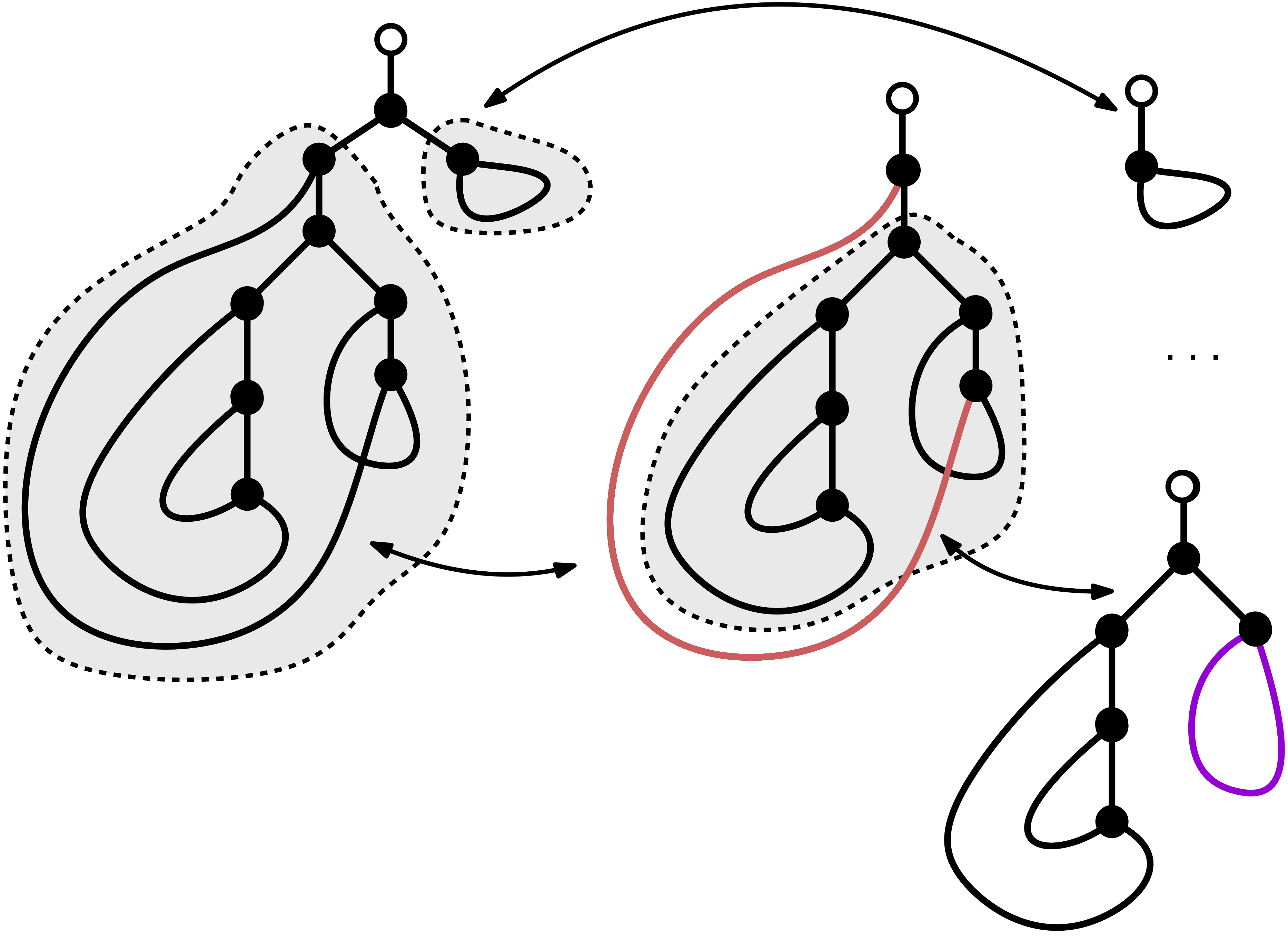

Deriving equations via guess-and-prove

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Schwinger-Dyson, elementary combinatorics

- Get one of the equations for free:

$$T(z, u) = uz + zT(z, u)^2 + z\partial_u T(z, u)$$

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$$T(z,u) = uz + zT(z,u)^2 + z\partial_u T(z,u)$$

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$$T(z,u) = uz + z^2 + zT(z,u)^2 + 2z^4\partial_z T(z,u)$$

- Use differential algebra to show equivalence of the two:

```
>
> eq2 := -L(u, z) + u*z + z*L(u, z)^2 + z*diff(L(u, z), u);
  eq1 := -L(u, z) + u*z + z^2 + z*L(u, z)^2 + 2*z^4*diff(L(u, z), z);
```

$$eq2 := -L(u,z) + uz + zL(u,z)^2 + z\left(\frac{\partial}{\partial u}L(u,z)\right)$$

$$eq1 := -L(u,z) + uz + z^2 + zL(u,z)^2 + 2z^4\left(\frac{\partial}{\partial z}L(u,z)\right) \quad (1)$$

```
>
> with(DifferentialAlgebra):
> R := DifferentialRing(blocks=[L, E], derivations=[z, u]):
> G := RosenfeldGroebner([eq2, eq1-E(u, z)], R);
```

$$G := [regular\_differential\_chain, regular\_differential\_chain] \quad (2)$$

```
> Equations(G[1])[2];
```

$$4\left(\frac{\partial^2}{\partial z \partial u}E(u,z)\right)E(u,z)z^5 - 4\left(\frac{\partial}{\partial z}E(u,z)\right)\left(\frac{\partial}{\partial u}E(u,z)\right)z^5 - \left(\frac{\partial}{\partial u}E(u,z)\right)^2 z^2 + 4E(u,z)^3 z - 4E(u,z)^2 z^2 u + E(u,z)^2 \quad (3)$$

```
> Equations(G[2]);
```

$$\left[-L(u,z) + uz + z^2 + zL(u,z)^2 + 2z^4\left(\frac{\partial}{\partial z}L(u,z)\right), -L(u,z) + uz + zL(u,z)^2 + z\left(\frac{\partial}{\partial u}L(u,z)\right), E(u,z)\right] \quad (4)$$

```
> BelongsTo(E(u, z), G[2]);
```

true (5)

```
> H := RosenfeldGroebner([eq1, eq2-E(u, z)], R):
> BelongsTo(E(u, z), H[2]);
```

true (6)

Proof due to Pierre Lairez.

A persistent phenomenon

A persistent phenomenon

easy to derive

- Loops in trivalent maps:

$$T(z,u,v) = uz + vz^2 + zT(z,u,v)^2 + z\partial_u(T(z,u,v) - uz)$$

$$T(z,u,v) = uz + (v-1)z^2 + zT(z,u,v)^2 + z\partial_v T(z,u,v)$$

$$T(z,0,v) = vz^2 + 2(v-1)^2z^5 + zT(z,0,v)^2 + 2z^4\partial_z T(z,0,v)$$
$$-2z^3(v-1)(T(z,0,v) - zT(z,0,v)^2)$$

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First two can be proven equivalent via diff. alg.
All three can be proven equivalent combinatorially (at $u = 0$).

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- Similar situations for bridges:

easy to derive

$$T(z,u,w) = uz + z(T(z,u,w)^2 + (v-1)T(z,u,w)^2) + z(\partial_u T(z,u,w)$$
$$+(v-1)\partial_u T(z,u,w))$$

easy to guess

$$\partial_w T(z,0,w) = -\frac{w^2T(z,0,w)^3 + z^2T(z,0,w) - T(z,0,w)^2}{(w^3-w^2)zT(z,0,w)^2 + wz^2 - (w-1)T(z,0,w)}$$

Combinatorics shows that two are equivalent.

Questions

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- Can this be done consistently and automatically?

Tetravalent maps

$$\int e^{-\left(\frac{\phi^2}{2}+z\frac{\phi^4}{4}\right)+J\phi} d\phi$$

easy to derive (SD)

$$\Phi = -z\partial_J^2\Phi - 3z\Phi\partial_J\Phi - z\Phi + J, \Phi = \langle\phi\rangle_J$$

easy to guess

birooted

$$F = \frac{1+zF^2+4z^2\partial_z F}{1-2z}, F = \partial_J \Phi_{J=0}$$

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$$T = u + zT^2 + z\partial_u T \to T|_{u=0} = z^2 + z(T|_{u=0})^2 + 2z^4\partial_z T|_{u=0}$$

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$$\int e^{-\left(\frac{\phi^2}{2}+z\frac{\psi^2}{2}+\frac{\phi^3}{3}+\frac{\psi^3}{3}+\phi\psi\right)+J\phi} d\phi d\psi$$

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$$A = -u - z(A^2 + \partial_u A) + aB(z,u,v)$$
$$B = -u - z(B^2 + \partial_u B) + aA(z,u,v)$$

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- Ubiquity of Riccati equations?

See: R. J. Martin and M. J. Kearney, "An exactly solvable self-convolutive recurrence", Aequationes mathematicae vol. 80, 2010

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Thanks!

Bibliography

[B75] Bender, E. A. (1975).
An asymptotic expansion for the coefficients of some formal power series.
Journal of the London Mathematical Society, 2(3), 451-458.

[CLP78] Cvitanović, P., Lautrup, B., & Pearson, R. B. (1978).
Number and weights of Feynman diagrams.
Physical review D, 18(6), 1939.

[AJW95] Ambjørn, Jan, J. Jurkiewicz, and Y. Watabiki (1995).
On the fractal structure of two-dimensional quantum gravity.
Nuclear Physics B, 454(1-2), 313-342.

[AL98] Ambjørn, J., & Loll, R. (1998).
Non-perturbative Lorentzian quantum gravity, causality and topology change.
Nuclear Physics B, 536(1-2), 407-434.

Bibliography

[BGGJ13] Bodini, O., Gardy, D., Gittenberger, B., & Jacquot, A. (2013).
Enumeration of Generalized BCI Lambda-terms.
The Electronic Journal of Combinatorics, P30-P30.

[HMR16] He, Y. H., McKay, J., & Read, J. (2013).
Modular subgroups, dessins d'enfants and elliptic K3 surfaces.
LMS Journal of Computation and Mathematics, 16, 271-318.

[Z16] Zeilberger, N. (2016).
Linear lambda terms as invariants of rooted trivalent maps.
Journal of functional programming, 26.

[BSZ21] Bodini, O., Singh, A., & Zeilberger, N. (2021).
Asymptotic distribution of parameters in trivalent maps and linear lambda terms.
Submitted, prepreint: arXiv preprint arXiv:2106.08291.

Bibliography

[S22] Singh, A. (2022).
Une approche unifiée de la combinatoire du lambda-calcul et des cartes: bijections et propriétés limites.
PhD diss., Université Paris-Nord-Paris XIII, 2022.